# Effective Adjunction Theory

Marco Andreatta

Dipartimento di Matematica
Universitá di Trento

Roma - 2019

# Introduction

Let $X$ be a projective variety over the complex field $\mathbb{C}$.
We assume that $X$ has mild singularities, namely

- **i)** it is normal, therefore we can define a *canonical Weil divisor* $K_X$
- **ii)** it has at most *canonical singularities*, i.e. $K_X$ is $\mathbb{Q}$-Cartier, and $\nu_* \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}) = \mathcal{O}_X(mK_X)$ for one (or for any) resolution of the singularities $\nu : \tilde{X} \to X$

In the category of projective spaces with canonical singularities the global sections of adjoint bundles (or of pluri-canonical bundles) are birational invariants:

**Lemma**

*Let $\pi : Y \to X$ be a birational morphism between projective varieties with at most canonical singularities, let L be a Cartier divisor on $X$ and let $a, b \in \mathbb{N}$. Then*

$$H^0(X, aK_X + bL) = H^0(Y, aK_Y + b\pi^*(L)).$$

# Uniruled

## Definition

A projective variety $X$ of dimension $n$ is said to be uniruled (respectively ruled) if there exists a projective variety $Y$ of dimension $n-1$ and a dominant rational (respectively birational) map $\varphi : \mathbb{P}^1 \times Y \cdots \to X$.

## Remark

*To be uniruled is obviously a birational property.*

## Proposition

*If X is uniruled with canonical singularities then*

$$H^0(X, mK_X) = 0 \ \textit{for} \quad \textit{all} \quad m > 0$$

*(if this is the case we say that X has Kodaira dimension minus infinity (or simply negative), i.e.* $k(X) = -\infty$*).*

# Conjecture of Mori

A long lasting question, stated by Mori in '85, is whether the converse is true:

**Conjecture**

*Let $X$ be a projective variety with canonical singularities, if $k(X) = -\infty$ then $X$ is uniruled.*

It is false for general singularities, for instance for $\mathbb{Q}$-Gorenstein rational, as some examples of J. Kollár show
(rational varieties with ample canonical divisor).

# Conjecture of Mumford

## Conjecture

*Let $X$ be a smooth projective variety;*
*if $H^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for all $m > 0$ then $X$ is rationally connected.*

J. Harris: ”Mori’s conjecture is well founded in birational geometry. Mumford’s seems to be some strange guess, how did he come up with that?”.

Mori’s implies Mumford’s:
via MRC fibration - Campana and Kollar-Mori-Miyaoka
and the Fibration theorem - Graber-Harris-Mazur-Starr.

# Abundance Conjecture



A famous related conjecture:

**Conjecture**

*Let $X$ be a projective variety with canonical singularities, if $K_X$ is nef then $|mK_X|$ is base point free for $m >> 0$.*

# Kleiman - Mori - ... cones

Let $X$ be a normal complex projective variety of dimension $n$.
We denote by $Div(X)$ the group of all Cartier divisors on $X$ and by $Num(X)$ the subgroup of numerically trivial divisors. The quotient group $N^1(X) = Div(X)/Num(X)$ is the Neron-Severi group of $X$.

In the vector space $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$, whose dimension is $\rho(X) := rkN^1(X)$, we consider some convex cones.

**(a)** $Amp(X) \subset N^1(X)_{\mathbb{R}}$ the convex cone of all *ample* $\mathbb{R}$-divisor classes; it is an open convex cone

**(b)** $Big(X) \subset N^1(X)_{\mathbb{R}}$ the convex cone of all *big* $\mathbb{R}$-divisor classes; it is an open convex cone

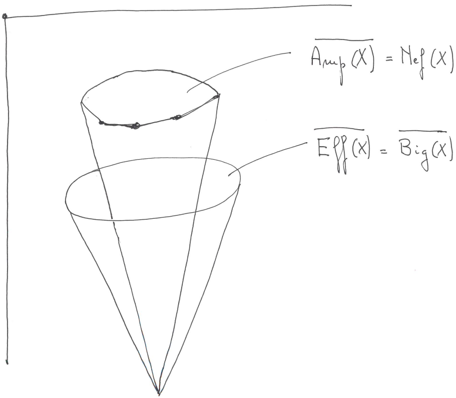
**(e)** $Eff(X) \subset N^1(X)_{\mathbb{R}}$ the convex cone spanned by the classes of all *effective* $\mathbb{R}$-divisors

**(n)** $Nef(X) = \overline{Amp(X)} \subset N^1(X)_{\mathbb{R}}$ the closed convex cone of all *nef* $\mathbb{R}$-divisor classes

**(p)** $\overline{Eff(X)} = \overline{Big(X)} \subset N^1(X)_{\mathbb{R}}$ **the closed convex cone of all pseudo-effective $\mathbb{R}$-divisor classes**

# ... cones

$$N^1(X)_{\mathbb{R}} = \frac{Div(X)}{Num(X)} \otimes \mathbb{R}$$



# .... cones

The above definitions actually lean on some fundamental results like the openess of the ample and big cones, the facts that $int\{\overline{Eff(X)}\} = Big(X)$ and $Nef(X) = \overline{Amp(X)}$.

Note that $Amp(X) \subset Nef(X) \cap Big(X)$ and that there are no inclusions between $Nef(X)$ and $Big(X)$.

Note also that if $\pi : X' \to X$ is a birational morphism and $D$ is a Cartier divisor on $X$ then $D$ is big (resp. pseudo-effective) if and only if $\pi^* D$ is big (resp. pseudo-effective).

# Remarks of Mori



## Remark

*If $K_X$ is not pseudo-effective, i.e. $K_X \notin \overline{Eff(X)}$, then $K_X \notin Eff(X)$, in particular $k(X) = -\infty$*

*If $X$ has canonical singularities "$K_X$ is (or is not) pseudo-effective" is a birational invariant*
*In particular $X$ uniruled implies that $K_X$ is not pseudo-effetive (this is the case for $K_{\mathbb{P}^1} \times K_Y$).*

Uniruledness $\Longrightarrow$ $K_X$ not pseudo-effective $\Longrightarrow$ $k(X) = -\infty$

# (Not) Pseudoeffectivity of the canonical bundle

Actually we have

**Theorem**

*Let $X$ be a projective variety with canonical singularities, if $K_X$ is not pseudoeffective then $X$ is uniruled*

It has been conjectured by Mori in '85, then proved first by by BDPP and then by BCHM, using the bend and breaking theory of Mori

# Termination of Adjunction another condition in the middle



Uniruledness, i.e. $K_X$ not pseudo-effective

$\Longrightarrow$ *Termination of Adjunction*

$\Longrightarrow k(X) = -\infty$

*Termination of Adjunction*, a rather delicate notion since in the literature there are different meanings for such a property:

# Termination of Adjunction-A

**Definition (Termination of Adjunction-A)**

There is a (For all)

$$H \in int\{\overline{Eff(X)}\}$$

there exists $m_0 = m_0(H) > 0$, natural number, such that

$$m_0 K_X + H \notin \overline{Eff(X)}$$

- Uniruledness $\Longrightarrow$ ToA-A: follows trivially since $K_X \equiv \lim_{m\to\infty} \frac{mK_X+H}{m}$
- ToA-A $\Longrightarrow$ Uniruledness: proved by Mori

# Termination of Adjunction

**Definition (Termination of Adjunction-B)**

For all $H \in int\{\overline{Eff(X)}\}$ there exists $m_0 = m_0(H) > 0$, natural number, such that $H^0(m_0K_X + H) = 0$

**Definition (Termination of Adjunction-C)**

For all $H$ very ample there exists $m_0 = m_0(H) > 0$, natural number, such that $H^0(m_0K_X + H) = 0$

**Definition (Termination of Adjunction-D)**

For some $H \in int\{\overline{Eff(X)}\}$ for every $k > 0$ there exists $m_0 = m_0(k) > 0$, natural number, such that $H^0(m_0K_X + kH) = 0$

# Termination of Adjunction

## Theorem

*The four definition are equivalent and they are equivalent to the fact that $X$ is uniruled.*

## Remark

*Note that Mori in '85, suggests that in principle (U) could have been stronger then (C):*

**We say that $X$ is $\kappa$-uniruled if $K_X$ is not pseudo-effective. We note that $\kappa$-uniruledness is slightly stronger than saying that adjunction terminates, i.e. $H^0(X, mK_X + H) = 0$ for each very ample divisor $H$ and some $m = m(H) > 0$.**

# Mori's notes





Thus rationality and unirationality seem too delicate to use in classification theory at present, and the difference of rationality and ruledness seem too subtle. On the other hand, we have

(11.2.4) *Fano n-folds* (nonsingular projective $n$-folds $X$ with ample $-K_X$) are uniruled by Kollár, and the uniruledness is stable under small and global deformations by Fujiki and Levin [**Fk2**, **L1**]. We note that Fano 3-folds $X$ are classified by Iskovskih and Shokurov [**Is1**, **Is2**, **Sho1**, **Sho2**] if $B_2(X) = 1$ and by Mori and Mukai [**MrMuk**] if $B_2(X) \geq 2$.

Therefore, unlike the case of $\kappa = 0$, we do not have a precise conjecture (cf. (10.2)), and the best we can hope is the uniruledness of varieties with $\kappa = -\infty$. We will introduce another notion.

(11.3) Let $D$ be a Cartier (or $\mathbf{Q}$-Cartier) divisor on a projective variety $X$. We say that $D$ is *pseudoeffective* if the following equivalent conditions are satisfied:

(i) $D$ is a limit of effective $\mathbf{Q}$-divisors; that is, there exist sequences $\{D_i\}_{i\geq 1}$ and $\{n_i\}_{i\geq 1}$ of effective Cartier divisors and natural numbers such that $D \approx \lim_i D_i/n_i$, i.e., $(D \cdot C) = \lim_i (D_i \cdot C)/n_i$ for every irreducible curve $C$.

(ii) $\kappa(nD + H) \geq 0$ for all big Cartier divisors $H$ and all natural numbers $n$.

(iii) There is a big Cartier divisor $H$ such that $\kappa(nD + H) \geq 0$ for all natural numbers $n$.

Indeed (ii)$\Rightarrow$(iii) is obvious and (iii)$\Rightarrow$(i) follows from $D \approx \lim_n (nD + H)/n$. Assuming (i), let $H$ be a big divisor and $n > 0$. By (1.9), there is an ample $\mathbf{Q}$-divisor $\overline{H}$ on $X$ such that $H \succ \overline{H}$. Then there is $i \gg 0$ such that $n(D - D_i/n_i) + \overline{H}$ is ample by Kleiman's criterion [**K1**], whence $\kappa(nD+H) \geq \kappa(nD+\overline{H}) \geq \kappa(D_i) \geq 0$ (ii).

(11.4) Let $X$ be a projective $n$-fold with only canonical singularities. We say that $X$ is *$\kappa$-uniruled* if $K_X$ is not pseudoeffective. We note that $\kappa$-uniruledness is slightly stronger than saying that the *adjunction terminates*, i.e., $|nK_X + H| = \emptyset$ for each very ample divisor $H$ and some $n = n(H) > 0$. One can easily see that the $\kappa$-uniruledness is a birational property by the following.

(11.4.1) LEMMA. *Let $f\colon X \to Y$ be a generically finite surjective morphism of projective varieties with only canonical singularities. Then*

(i) *if $X$ is $\kappa$-uniruled then so is $Y$, and*

(ii) *the converse holds if $f$ is birational.*

Indeed (i) follows from $nK_X \succ f^*(nK_Y)$ for some $n \in \mathbf{N}$ (follows from the definition of the canonical singularity), and (ii) follows by $f_*\mathcal{O}(nK_X) = \mathcal{O}(nK_Y)$ for all $n > 0$.

(11.4.2) COROLLARY. *For a projective variety $X$ with only canonical singularities, we have*

$$X \text{ is uniruled} \Rightarrow X \text{ is } \kappa\text{-uniruled} \Rightarrow \kappa(X) = -\infty.$$

Indeed the second implication is obvious by the definition, and for the first, it is enough to show $\mathbf{P}^1 \times Y$ is $\kappa$-uniruled by (11.4.1), which is an easy exercise.

# Siu's Lemma



The results follow by a fundamental result of Siu on pseudo-effective Cartier divisors.

**Lemma**

*Let $X$ be a smooth projective variety of dimension $n$, and let $H$ be a very ample divisor on $X$ and $G := (n+1)H + K_X$.*
*For every pseudo-effective divisor $F$ on $X$ we have $H^0(X, F+G) \neq 0$.*

# Sketch of proof of the equivalence

(C): For some big Cartier divisor $H$ we have $H^0(X, m_0K_X + kH) = 0$ for every $k > 0$ and some $m_0 = m_0(k) > 0$

Assume by contradiction that $X$ is not uniruled, i.e. $K_X$ is pseudo-effective (and for simplicity that $X$ is smooth).

Let $H$ be any big Cartier divisor on $X$; we have $lH = A + N$ with $A$ ample and $N$ effective for some $l > 0$, thus $hlH = hA + hN$ with $hA$ very ample for some $h > 0$.

Hence for every $m_0 > 0$ we have
$\dim H^0(X, m_0K_X + (n+1)hlH) =$
$\dim H^0(X, (m_0 - 1)K_X + (K_X + (n+1)hA) + (n+1)hN) \geq$
$\dim H^0(X, (m_0 - 1)K_X + (K_X + (n+1)hA))$.

By the Lemma the last is positive, contradicting (C).

# proof of the Lemma



The proof of the Lemma is by induction on $n$; obvious for $n = 1$.

Let $D \in |H|$ be a general divisor, by Bertini theorem smooth variety of dimension $n-1$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(K_X+nH+F) \to \mathcal{O}_X(K_X+(n+1)H+F) \to \mathcal{O}_D(K_D+nH_D+F_D) \to 0$$

If $H^1(X, K_X + nH + F) = 0$ then we have the exact sequence

$$0 \to H^0(X, K_X + nH + F) \to H^0(X, K_X + (n+1)H + F) \to$$
$$\to H^0(D, K_D + nH_D + F_D) \to 0.$$

and we conclude by induction.

# Vanishing Theorem



**Theorem**

*(Vanishing Theorem: Kawamata - Viehweg - Nadel) Let $X$ be a smooth projective variety (or with mild singularities) and let $L$ be a nef and big divisor and $F$ any effective or pseudo-effective $\mathbb{Q}$-divisor.*
*Then $H^i(X, K_X + L + F + \mathcal{I}(F)) = 0$ for $i > 0$*

The problem is to define $\mathcal{I}(F)$ as a coherent ideal sheaf for effective (Kawamata - Viehweg) or for pseudoeffective (Nagel - Siu).

# Classical Termination

**Definition (Termination of Adjunction in the classical sense)**

Let $X$ be a normal projective variety and let $H$ be an *effective Cartier divisor* on $X$ (or very ample)
*Adjunction Terminates in the classical sense* for $H$ if there exists an integer $m_0 \geq 1$ such that

$$H^0(X, mK_X + H) = 0$$

for every integer $m \geq m_0$.

If $X$ is a projective variety with canonical singularities:

Uniruledness $\Longrightarrow$ Adjunction Terminates CS for $H \Longrightarrow k(X) = -\infty$.

# Classical Termination

We conjecture that this more general definition is actually equivalent to the previous ones. We actually hope to prove:

**Conjecture**

*Let $X$ be a projective variety of dimension $n$ with canonical singularities. Assume that $X$ has no extremal rays whose associated contraction is of birational type.*
*If for some big and nef divisor $H$ Adjunction Terminates in the classical sense, then $X$ is uniruled.*

It is true in dimension two; together with the MMP for surface (*superficie adeguatamente preparate*) and the invariance for birational map of sections of adjoint systems provide a modern proof of the Castelnuovo-Enriques Theorem which says that Termination of Adjunction holds for a very ample line bundle on a surface $S$ if and only if $S$ is ruled.

# ”Special” Termination

## Proposition

*Let $X$ be a smooth projective variety of dimension $n$ and let $H$ be a very ample divisor on $X$ which is of the form $H = (n+1)L$.*
*If $H^0(X, mK_X + H) = 0$ for some natural number $m \geq 1$, then $X$ is uniruled.*

This is a different way to consider (effective) termination of adjunction. It follows as a straightforward consequence of Siu’s Lemma and the main result in BDPP.

# Quasi polarized pair

A *quasi polarized pair* is a pair $(X, L)$ where $X$ is a projective varieties with at most $\mathbb{Q}$-factorial terminal singularities and $L$ is a nef and and big Cartier divisor on $X$.
If $L$ is ample we call the pair $(X, L)$ a *polarized pair*.

Let

$$\sigma(X, L) := \sup\{s \in \mathbb{R} : sK_X + L \in \overline{\mathit{Eff}(X)}\}$$

the effective log threshold, and

$$\nu(X, L) := \inf\{t \in \mathbb{R} : K_X + tL \in \overline{\mathit{Eff}(X)}\}$$

the (unnormalized) spectral value
$(-\nu(X, L)$ is called the Kodaira energy of the pair $(X, L))$.

A classification of polarized pairs with high spectral value was started long ago by Fujita and Sommese.

# Bounds for the threshold

## Remark

*If $K_X$ is not pseudo-effective then $\nu(X,L) > 0$.*

*Using Hilbert polynomial (degree $n$) and the vanishing theorem one can prove that $\nu(X,L) \leq n+1$.*
*Equality holds if and only if $(X,L)$ is birational equivalent, via a MMP with scaling given by $L$ (called $0$-reduction), to the pair $(\mathbb{P}^n, \mathcal{O}(1))$*

# Two basic results

The two following results are in [BCHM].

**Theorem**

*If $K_X$ is not pseudo-effective and $L$ is ample then $\sigma(X,L) = 1/\nu(X,L) > 0$ are rational numbers.*

**Theorem**

*Let $(X,L)$ be a quasi polarized pair and $t > 0$. If $K_X + tL \in \overline{Eff(X)}$, then there exists $N \in \mathbb{N}$ such that $H^0(X, N(K_X + tL)) \neq 0$.*

Note that for $t = 0$ the statement of the Proposition, together with MMP, would amount to Abundance Conjecture.

# Effectivity of non vanishing

The next Conjecture is an effective version of the above Proposition.

## Conjecture

*Let $(X, L)$ be a quasi polarized pair and $t > 0$. If $K_X + tL \in \overline{\mathit{Eff}(X)}$, then $H^0(X, K_X + tL) \neq 0$.*

For $t = 1$ this is a version of the so-called Ambro-Ionescu-Kawamata conjecture, which is true for $n \leq 3$ .
For $t = n - 1$ this is a conjecture by Beltrametti and Sommese.

We considered the following milder conjecture, which clearly implies the above one if $L$ is effective.

## Conjecture

*Let $(X, L)$ be a quasi polarized pair of dimension $n$. Then $H^0(X, K_X + tL) = 0$ for every integer $t$ with $1 \leq t \leq s$ if and only if $K_X + sL$ is not pseudo-effective.*

# Some Results

The Conjecture is true for $s = n$; we actually show that this case happens if and only if the pair $(X, L)$ is birationally equivalent (via a 0-reduction) to the pair $(\mathbb{P}^n, \mathcal{O}(1))$.

For $s = n - 1$ the conjecture was essentially proved by Höring. We prove a slightly more explicit version of his result, namely, we show that this case happens if and only if the pair $(X, L)$ is birationally equivalent (via a 1-reduction) to a finite list of pairs.

If $s = n - 2$ the *if* part of the Conjecture is true.

The case $n = 4$ is true (Fukuma and some generalizations).

# Fujita-Sommese theory of the reductions Minimal Model Program with scaling



The above results can be proved via the Theory of Reductions, started by Fujita and Sommese and described in the Beltrametti-Sommese's book.

Nowadays it can be expressed via the Minimal Model Program with scaling by Birkar-Cascini-Hacon-McKernan

# Minimal Model Program- BCHM



By BCHM on a klt log pair $(X, \Delta)$, with $\Delta$ big, we can run a

$K_X + \Delta$- Minimal Model Program with scaling:

$$(X_0, \Delta_0) = (X, \Delta) \to (X_1, \Delta_1) \to - - -- \to (X_s, \Delta_s)$$

such that:

1) $(X_i, \Delta_i)$ is a klt log pair, for $i = 0, ..., s$;

2) $\varphi_i : X_i \to X_{i+1}$ is a birational map which is either a divisorial contraction or a flip associated with an extremal ray $R_i = \mathbb{R}^+[C_i]$ such that $(K_{X_i} + \Delta_i) \cdot C_i < 0$

3)

- either $K_{X_s} + \Delta_s$ is nef (i.e. $(X_s, \Delta_s)$ is a log Minimal Model),
- or $X_s \to Z$ is a Mori fiber space relatively to $K_{X_s} + \Delta_s$

(depending on the pseudeffectivity of $K_X + \Delta$).

# MMP for a q.p. pair

Let $(X, L)$ be a quasi-polarized variety and let $r \in \mathbb{Q}^+$.

Lemma (zip L into a boundary). Since $L$ is nef and big there exists an effective $\mathbb{Q}$-divisor $\Delta^r$ on $X$ such that

$$rL \sim_{\mathbb{Q}} \Delta^r \quad \text{and} \quad (X, \Delta^r) \text{ is Kawamata log terminal.}$$

Run a $K_X + \Delta^r$-MMP and get a birational klt pair $(X_s, \Delta^r_s)$ which is
- either a Minimal Model ($K_{X_s} + \Delta_s$ is nef)
- or $X_s \to Z$ is a Mori fiber space relatively to $K_{X_s} + \Delta_s$.

Remarks/Problems

- $(X_s, \Delta^r_s)$ is not necessarily an $(r)$ q.p. pair, i.e. we do not have a priori a nef and big Cartier divisor $L_s$ such that $rL_s \sim_{\mathbb{Q}} \Delta^r_s$.
- Beyond the existence of the MMP, it would be nice to have a "description" of each steps and eventually of the Mori fiber spaces.

# Description of the contractions

**Theorem**

*Let $\varphi : X \to Y$ be a birational contraction associated with an extremal ray $R = \mathbb{R}^+[C]$ on a q.p. pair, such that $L \cdot C > 0$ and $\tau > (n-3)$. (These are the birational maps in a $K_X + \Delta^{n-3}$-MMP).*

*If it is divisorial then $\varphi$ is a weighted blow-up of a particular cyclic quotient or hyperquotient singularities*

(This is a "lifting" in higher dimension of the classification in dimension 3 by Mori, Kawamata and Kawakita)

It is small then $\varphi$ is a Francia-Mori flip ....???