Families of Rational Curves which determine the structure of the (projective) Space

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Hilbert Scheme

Let X and Y be a normal projective schemes (of f. t. over $k = \bar{k}$).
$\mathrm{Hilb}(X)$: the Hilbert scheme of proper subschemes of X.
$\mathrm{Hom}(Y, X)$: open subscheme $\subset \mathrm{Hilb}(X \times Y)$ of morphisms from Y to X.
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Theorem

Let $f : Y \longrightarrow X$ be a morphism. Assume that Y is without embedded points, that X has no embedded points contained in $f(Y)$ and the image of every irreducible component of Y intersect the smooth locus of X. Then

- *The tangent space of $Hom(Y, X)$ at $[f]$ is naturally isomorphic to*

$$Hom_Y(f^*\Omega_X{}^1, \mathcal{O}_Y).$$

- *The dimension of every irreducible component of $Hom(Y, X)$ at $[f]$ is at least*

$$dimHom_Y(f^*\Omega_X{}^1, \mathcal{O}_Y) - dimExt_Y{}^1(f^*\Omega_X{}^1, \mathcal{O}_Y).$$

Hilbert Scheme

Let $f : C \longrightarrow X$ be a morphism from a proper curve to a scheme; L a line bundle on X.
We define the intersection number of C and L as:

$$C \cdot L := deg_C f^* L$$

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In the special case of the Hilbert scheme of curves, thank to Riemann Roch theorem, we have the following nice result.

Theorem

Let C be a proper algebraic curve without embedded points and $f : C \longrightarrow X$ a morphism to a smooth variety X of pure dimension n. Then

$$dim_{[f]} Hom(C, X) \geq -K_X \cdot C + n\chi(\mathcal{O}_C).$$

Moreover equality holds if $H^1(C, f^* T_X) = 0$.

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Proof If F is a locally free sheaf on a scheme Z, then

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Thus

$$dim_{[f]}\mathrm{Hom}(C, X) \geq h^0(C, f^*T_X) - h^1(C, f^*T_X),$$

which, by Riemann -Roch, is equal to

$$\chi(C, f^*T_X) = degf^*T_X + n\chi(\mathcal{O}_C) = -K_X{}^{\cdot}C + n\chi(\mathcal{O}_C).$$

Existence of Rational Curves

A *rational curve* on X is a non constant morphism $\mathbb{P}^1 \longrightarrow X$.

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The following is a fundamental result of S. Mori.

Theorem

Let X be a smooth projective variety over an algebraically closed field (of any characteristic), C a smooth, projective and irreducible curve and $f : C \longrightarrow X$ a morphism. Assume that

$$-K_X \cdot C > 0.$$

Then for every $x \in f(C)$ there is a rational curve $D_x \subset X$ containing x and such for any nef $\mathbb{R}$-divisor L :

$$L \cdot D_x \leq 2 dim X \left(\frac{L \cdot C}{-K_X \cdot C}\right) \quad \text{and} \quad -K_X \cdot D_x \leq dim X + 1.$$

Proof: step 1

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Step 1. We have

$$dim_{[f]}\text{Hom}(C,X) \geq -K_X{}^{\cdot}C + n(1-g).$$

Take $x = f(0) \in f(C)$; since n conditions are required to fix the image of the basepoint 0 under f, morphisms f of C into X sending 0 to x have a deformation space of dimension

$$\geq -K_X{}^{\cdot}C + n(1-g) - n = -K_X{}^{\cdot}C - ng.$$

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If this quantity is positive there must be a non-trivial one-parameter family of deformations of the map f keeping the image of 0 fixed.
In particular, we can find a nonsingular (affine) curve D and a morphism (evaluation) $g : C \times D \to X$, thought of as a nonconstant family of maps, *all sending* 0 *to the same point* x.

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Step 2. We argue now that D cannot be complete (Bend and Break). In fact otherwise consider U, a neighborhood of x in C and the projection map $\pi : U \times D \to U$.

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Then π is a **proper**, surjective morphism with **connected fiber of dimension** 1. Moreover $g(\pi^{-1}(x))$ is a single point.

By the **Rigidity Lemma**, $g(\pi^{-1}(y)$ is a single point for all y in U, i.e. the family would have to be constant.

Proof: step 2

Proof of the rigidity Lemma

Let $W = im(\pi \times g) \subset U \times X$ and consider the proper morphisms

$$\pi : U \times D \to W \to U,$$

where the first map, $h = U \times D \to W$, is defined by $h(t) = (\pi(t), g(t))$ and the second, $p : W \to U$, is the projection to the first factor.

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For any $w \in W$, $h^{-1}(w) \subset \pi^{-1}(p(w)$, dim $h^{-1}(w) \geq 1$ and dim $\pi^{-1}(p(w)) = 1$. Therefore $h^{-1}(w)$ is a union of irreducible components of $\pi^{-1}(p(w))$, and so $h(\pi^{-1}(p(w))) = p^{-1}(p(w))$ is finite.

It is a single point since $\pi^{-1}(p(w))$ is connected.

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Blow up a finite number of points to resolve the undefined points to get $Y \to C \times \overline{D}$ whose composition given by $\pi : Y \to X$ is a morphism. Let $E \subset Y$ be the exceptional curve of the last blow up.

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Since it was actually needed, it can't be collapsed to a point, and hence $\pi(E)$ is our desired rational curve.

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$F_p : C \to C$ is set-theoretically injective but it is an endomorphism of degree p. Take $h = F_p^r \circ f$ and call C' and C'' the curves respectively given as image of f and of h.

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For r high enough $-C'' \cdot K_X \geq ng + 1$ and therefore

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In this way we prove the existence of a rational curve through x for almost all $p > 0$.

Proof: step 5

Step 5. Algebra.

Principle. If a homogeneous system of algebraic equations with integral coefficients has a non trivial solution in $\overline{\mathbb{F}}_p$ for infinitely many p, then it has a non trivial solution in any algebraically closed field.

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A map $\mathbb{P}^1 \to X \subset \mathbb{P}^N$ of limited degree with respect to $-K_X$ can be given by a system of equations. Since this system has a non trivial solution for infinitely many p, it has a solution in any algebraically closed field by the above principle.

Miyaoka test for uniruledness

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The following Theorem of Y. Miyaoka, which generalizes the Mori's result, is a powerful uniruledness criteria.

Theorem

Let X be a smooth and proper variety over $\mathbb{C}$. Then X is uniruled if and only if there is a quotient sheaf $\Omega_X{}^1 \longrightarrow F$ and a family of curves $\{C_t\}$ covering an open subset of X such that $F_{|C_t}$ is locally free and $deg(F_{|C_t}) < 0$ for every t.

Families of Rational Curves

Let $\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the open suscheme corresponding to those morphisms $f : \mathbb{P}^1 \longrightarrow X$ which are birational onto their image, that is f is an immersion at its generic point. This is an open condition.

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If $f : \mathbb{P}^1 \longrightarrow X$ is any morphism and $h \in Aut(\mathbb{P}^1)$, then $f \circ h$ is "counted" as a different morphism.

The group $Aut(\mathbb{P}^1)$ acts on $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ and it is the quotient that "really parametrizes" morphisms of $\mathbb{P}^1$ into X. It can be proved that the quotient exists (Mori-Mumford-Fogarty) ; its normalization will be denoted $RatCurves^n(X)$ and called the *space of rational curve on* X.

Rational Curves

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Special families of Rational Curves

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Given a point $x \in X$, one can similarly find a scheme $\mathrm{Hom}_{bir}(\mathbb{P}^1, X, x)$ whose geometric points correspond to generically injective morphisms from $\mathbb{P}^1$ to X which map the point $[0 : 1]$ to x. The quotient, in the sense of Mumford, by the group of automorphism of $\mathbb{P}^1$ which fixes the point $[0 : 1]$, will be denoted by $RatCurves^n(x, X)$ and called the *space of rational curves through* x.

Families of Rational Curves

We obtain a diagram as follows

$$\begin{tikzcd} \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \arrow[r, "U"] \arrow[d] & Univ(X) \arrow[r, "i"] \arrow[d, "\pi"'] & X \\ \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \arrow[r, "u"] & RatCurves^n(X) & \end{tikzcd} \tag{2.0.1}$$

where U and u have the structure of principal $Aut(\mathbb{P}^1)$-bundle; π is a $\mathbb{P}^1$-bundle. The restriction of i to any fiber of π is generically injective, i.e. birational onto its image.

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Similarly for a given point $x \in X$:

$$\begin{tikzcd}
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, x) \times \mathbb{P}^1 \arrow[r, "U"] \arrow[d] & Univ(x, X) \arrow[r, "i_x"] \arrow[d, "\pi"'] & X \\
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, x) \arrow[r, "u"] & RatCurves^n(x, X) &
\end{tikzcd} \tag{2.0.2}$$

Families of Rational Curves

Let $B = \emptyset$ or x.

Let $F : \mathbb{P}^1 \times Hom^n_{bir}(\mathbb{P}^1, X, B) \to X$ be the *universal map* defined by $F(f, p) = f(p)$; F is the composition $i_B \circ U$.

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Let $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$, be an irreducible component and V_x be the set of elements in V passing through $x \in X$.

We denote $Locus(V) := F(V \times \mathbb{P}^1)$ and $Locus(V_x) := F(V_x \times \mathbb{P}^1)$.

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Let $f : \mathbb{P}^1 \to X \in Hom^n_{bir}(\mathbb{P}^1, X, B)$ and assume X is smooth along $f(\mathbb{P}^1)$. Then the tangent space of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, B)$ at $[f]$ is naturally isomorphic to

$$T_{[f]}\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, B) = \mathrm{Hom}_Y(f^*\Omega^1_X(-B), \mathcal{O}_Y) = H^0(\mathbb{P}^1, f^*T_X(-B)).$$

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In particular the tangent map of F at the point (f, t) :

$$dF_{f,t} : H^0(\mathbb{P}^1, f^*T_X(-B)) \oplus T_{\mathbb{P}^1,t} \to T_{X,f(t)} \tag{2.0.3}$$

is given by

$$(\sigma, u) \to (df_t(u) + \sigma(t)).$$

Splitting of the Tangent

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Let X be a smooth projective variety of dimension n.
Let $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$, be an irreducible component; for a rational curve $f : \mathbb{P}^1 \longrightarrow X$ in V we therefore have

$$f^*TX = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n).$$

The splitting type, i.e. the a_i, are the same for all members $f \in V$.

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By the general theory of Hilbert scheme we have presented above, if $a_i \geq -1$ (resp. $a_i \geq 0$), then V (resp. V_x) is smooth at $[f]$ and $dimV = dimX + \Sigma a_i$.

Splitting of the Tangent

Moreover, since $dimLocus(V) = rk(dF)$ at a generic point $x \in X$, using the above description of the tangent map of F, we have :

$$dimLocus(V) = \sharp\{i : a_i \geq 0\}.$$

Similarly, for general $x \in X$:

$$dimLocus(V_x) = \sharp\{i : a_i \geq 1\}.$$

Special Families of R.C.

Definition

f is called **free** if $a_i \geq 0$ for every i. Equivalently f is free if f^*TX is generated by its global sections or if $H^1(\mathbb{P}^1, f^*T(X)(-1)) = 0$.

Special Families of R.C.

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From the above observations we have immediately the following.

Proposition

(Assume $char(k) = 0$). X is uniruled if and only through a general point $x \in X$ there is a free rational curve.

Special Families of R.C.

Remark

Note that X is uniruled if and only if there exists a family of rational curve $\mathcal{V}$ such that $i : Univ(X) \longrightarrow X$ is dominant.
(This follows from the fact that the irreducible components of $RatCurves^n(X)$ are numerable; which in turn follows from the fact that families of a given degree, with respect to a very ample line bundle, are finite, depending on the Hilbert polynomial).

In this case we call $\mathcal{V}$ a unruling for X.

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In this case we call $\mathcal{V}$ a unruling *for X.*

Theorem

Let $\mathcal{V}$ be an irreducible component of $RatCurves^n(X)$. Denote by $\mathcal{V}^{free} \subset \mathcal{V}$ the parameter space of members of $\mathcal{V}$ that are free. Then $\mathcal{V}$ is a uniruling if and only if $\mathcal{V}^{free}$ is nonempty. In this case, $\mathcal{V}^{free}$ is a Zariski open subset of the smooth locus of $\mathcal{V}$.

Special Families of R.C.

Given a unirulling $\mathcal{V}$ on X and a point $x \in X$, let $\mathcal{V}_x$ be the normalization of the subvariety of $\mathcal{V}$ parametrizing members of $\mathcal{V}$ passing through x. Since by the above Theorem non-free rational curves do not cover X, for general point $x \in X$, the structure of $\mathcal{V}_x$ is particularly nice:

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Theorem

For a uniruling $\mathcal{V}$ on a projective manifold X and a general point $x \in X$, all members of $\mathcal{V}_x$ belongs to $\mathcal{V}^{free}$. Furthermore, the variety $\mathcal{V}_x$ is a finite union of smooth quasi-projective varieties of dimension $\deg_{K_X^{-1}}(\mathcal{V}) - 2$.

Special Families: unbreakable

Definition

A family of rational curve $\mathcal{V}$ on a projective manifold X is **locally unsplit** or **unbreakable** if $\mathcal{V}_x$ is projective for a general $x \in X$. Members of an unbreakable uniruling on X will be called minimal rational curves on X.

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Definition

Let L be an ample line bundle on a projective manifold X. A uniruling $\mathcal{V}$ is **minimal** with respect to L, if $deg_L(\mathcal{V})$ is minimal among all unirulings of X. A uniruling is a minimal uniruling if it is minimal with respect to some ample line bundle.

Minimals are Unbreakables.

Proposition

Minimal unirulings exist on any uniruled projective manifold. A minimal uniruling is unbreakable. In particular there exist unbreakable unirulings on any uniruled projective manifold.

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Sketchy geometric proof: suppose for a uniruling $\mathcal{V}$, which is minimal with respect to an ample line bundle L, the variety $\mathcal{V}_x$ is not projective for a general point $x \in X$. Then the members of $\mathcal{V}_x$ degenerate to reducible curves all components of which are rational curves of smaller L-degree than the members of $\mathcal{V}$ and some components of which pass through x. Collecting those components passing through x, as x varies over the general points of X, gives rise to another uniruling $\mathcal{V}'$ satisfying $deg_L(\mathcal{V}') < deg_L(\mathcal{V})$, a contradiction to the minimality of $deg_L(\mathcal{V})$.

Property of Unbreakables

Mori shows that a weaker version of this property continues to hold for unbreakable unirulings:

Theorem

Let $\mathcal{V}$ be an unbreakable family. Then for a general point $x \in Locus(\mathcal{V})$ and any other point $y \in Locus(\mathcal{V}_x)$, there does not exist a positive-dimensional family of members of $\mathcal{V}$ that pass through both x and y. (This is the definition of generically unsplit family in [Ko] IV.2.1)

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The theorem is proved again by a ”bend-and-break” plus rigidity argument. Geometrically, it says that any 1-dimensional family of rational curves which share two distinct points in common must degenerate into a reducible curve. This is the most important geometric property of an unbreakable uniruling.

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If $\mathcal{V}$ is unbreakable and we let $V = u^{-1}(\mathcal{V})$ and $\Pi : V \to X \times X$ be the map $[f] \to (f(0), f(\infty))$, the Theorem says that the fiber of Π over the generic point of Im (Π) has dimension at most one.

Inequalities for Unbreakable Loci

Proposition

Let $\mathcal{V}$ be an unbreakable family and let $V = u^{-1}(\mathcal{V})$.
If $x \in X$ is a general point in $Locus(V)$, then

$$dim V = dim Locus(V) + dim Locus(V_x) + 1.$$

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Proof. By upper-semicontinuity, for $x \in Locus(V)$

$$dim\{[f] \in V : f(0) = x\} \geq dim V - dim Locus(V).$$

If $y \in Locus(V_x)$ similarly

$$dim\{[f] \in V : f(0) = x, f(\infty) = y\} \geq dim V - dim Locus(V) - dim(Locus(V_x)$$

equality holds for general x and y. The proposition follows since

$$1 = dim \Pi^{-1}(x, y) = dim\{[f] \in V : f(0) = x, f(\infty) = y\}.$$

Inequalities for Unbreakable Loci

Combining this Theorem with 2 we obtain the following result.

Corollary

Let $\mathcal{V}$ be an unbreakable family and let $V := u^{-1}(\mathcal{V})$. Then

- $dimX + deg_{-K}V \leq dimLocus(V) + dimLocus(V, 0 \to x) + 1$
- $dimX + deg_{-K}V \leq 2dimLocus(V) + 1 \leq 2dimX + 1$
- $deg_{-K}V \leq dimLocus(V, 0 \to x) + 1 \leq dimX + 1$

Special Families: Unbendings

Definition

A rational curve $C \subset X$ is **unbending** if under the normalization $\nu_C : \mathbb{P}^1 \longrightarrow C \subset X$, the vector bundle ${\nu_C}^* T(X)$ has the form

$$ {\nu_C}^* T(X) = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p} $$

for some integer p satisfying $0 \leq p \leq n-1$, where $n = dimX$.
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Remark

If $f : \mathbb{P}^1 \to C \subset X$ is an unbending member of $\mathcal{V}_x$ the differential $Tf : T(\mathbb{P}^1) \longrightarrow f^* T(X)$ is an isomorphism of $T(\mathbb{P}^1)$ and the unique $\mathcal{O}(2)$ summand. Therefore Tf_p is non zero at every $p \in \mathbb{P}^1$. Recall that a curve is immersed if its normalization has rank one at every point; therefore an unbending member is immersed.

Unbreakable are generically Unbending

The above definition allows an infinitesimal version of Theorem 11.

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Sketch of proof. Let $[f] \in V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ a general element of an unbreakable uniruling V
(i.e. $V = u^{-1}\mathcal{V}$ with $\mathcal{V}$ an unbreakable uniruling).
Let $f^*TX = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$; by assumption $a_i \geq 0$ for every i.

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Then
$dimX + \Sigma a_i = dimV = dimLocus(V) + dimLocus(V, 0 \to x) + 1 =$
$dimX + \sharp\{i : a_i \geq 1\} + 1.$

Therefore $\Sigma a_i = \sharp\{i : a_i \geq 1\} + 1$, that is at most one of the a_i is at least two.

Unbreakable are generically Immersed

Kebekus has carried out an analysis of singularities of members of $\mathcal{V}_x$ and proved that they are considerably well behaved. Among other things, he has shown

Theorem

For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, members of $\mathcal{V}_x$ which are singular are a finite number. Moreover the singular ones are immersed at the point corresponding to x.

Special Families: Unsplit

Definition

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Definition

A family of rational curve $\mathcal{V}$ on a projective manifold X is **unsplit** if $\mathcal{V}$ is projective.

Let $\mathcal{V}$ be an unsplit uniruling. It defines a relation of *rational connectedness with respect to* $\mathcal{V}$, which we shall call rc$\mathcal{V}$ relation for short, in the following way: $x_1, x_2 \in X$ are in the rc$\mathcal{V}$ relation if there exists a chain of rational curves parametrized by morphisms from $\mathcal{V}$ which joins x_1 and x_2. The rc$\mathcal{V}$ relation is an equivalence relation and its equivalence classes can be parametrized generically by an algebraic set.

Rc$\mathcal{V}$ fibration

We have the following result due to Campana and, independently, to Kollár-Miyaoka-Mori (see ([Ko], IV.4.16).

Theorem

There exist an open subset $X^0 \subset X$ and a proper surjective morphism with connected fibers $\varphi^0 : X^0 \to Z^0$ onto a normal variety, such that the fibers of φ^0 are equivalence classes of the rc$\mathcal{V}$ relation.

We shall call the morphism φ^0 an rc$\mathcal{V}$ fibration. If Z_0 is just a point then we will call X a **rationally connected manifold with the respect to the family** $\mathcal{V}$, in short an rc$\mathcal{V}$ manifold.

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Lemma

Let X be a manifold which is rationally connected with the respect to a unsplit uniruling $\mathcal{V}$. Then $\rho(X) := dimN_1(X) = 1$ and X is a Fano manifold.

Also in this case the proof is a sort of an (easy) bend and break lemma.