Families of Rational Curves which determine the structure of the (projective) Space

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The Tangent Map

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Definition

The rational map $\Phi_x : \mathcal{V}_x - - - - > P(T_xX)$, defined, at $[f] \in \mathcal{V}_x$ which is smooth at 0 , by

$$\Phi_x([f]) = [(Tf)_0(\partial/\partial t)],$$

is called the tangent map (c.f. [Mori79, pp.602-603]).
It sends a member of $\mathcal{V}_x$ which is smooth at 0 to its tangent direction.

Notation. By P we denote the "natural projectivisation".
With t we denote a local coordinate around $0 \in \mathbb{P}^1$.

The Tangent Map

Proposition

If $f : \mathbb{P}^1 \to C \subset X$ *is an unbending member of* $\mathcal{V}_x$, *the tangent map can be extended to* $[f]$, *even when* C *is singular at* x, *because the differential* $Tf : T(\mathbb{P}^1) \longrightarrow f^*T(X)$ *is injective.*
Moreover Φ_x *is immersive at* $[f] \in \mathcal{V}_x$.

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Moreover Φ_x is immersive at $[f] \in \mathcal{V}_x$.

In particular for an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, the tangent map Φ_x is generically finite over its image.

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Let $V = u^{-1}\mathcal{V}$ the Hilbert family corresponding to $\mathcal{V}$, $B = \emptyset$ or x:
$T_{[f]}V_B = H^0(\mathbb{P}^1, f^*T_X(-B)) = H^0(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}(-B))$.
Passing to the quotient by $Aut(\mathbb{P}^1)$, i.e. passing to $\mathcal{V}$, we delete the part corresponding to $T(\mathbb{P}^1)$:

$$T_{[f]}(\mathcal{V}_x) = H^0(\oplus\mathcal{O}^p \oplus \mathcal{O}(-1)^{n-1-p}) \subset T_{[f]}(\mathcal{V}) = H^0(\oplus\mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}).$$

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Take $v \in T_{[f]}(\mathcal{V}_x) \subset T_{[f]}(\mathcal{V})$; we can find a deformation f_t of $f_0 := f$ such that $\frac{df}{dt}_{|t=0} = v$. Let z be a local coordinate in $\mathbb{P}^1$ centered at 0.

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Then the differential $d\Phi_x : T_{[f]}(\mathcal{V}_x) \to T_{\Phi_x([f])}P(T_xX)$ send v to

$$d\Phi_x(v) = \frac{d}{dt}_{|t=0} \frac{df_t}{dz}_{|z=0} = \frac{d}{dz}_{|z=0} \frac{df_t}{dt}_{|z=0} = \frac{dv}{dz}_{|z=0}.$$

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To derive v with respect to z we think it in
$T_{[f]}(\mathcal{V}) = H^0(\oplus\mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p})$; a non zero section here has non vanishing differential.

The Tangent Map

Using the above mentioned result of Kebekus one can prove the following.

Theorem

For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, the tangent morphism $\Phi_x : \mathcal{V}_x \longrightarrow P(T_x X)$ can be defined by assigning to each member C of $\mathcal{V}_x$ its tangent direction.
This morphism Φ_x is finite over its image.

The Tangent Map

Proof. Let $i_x : U_x \to X$ be the evaluation map; by Kebekus the preimage $i_x^{-1}(x)$ contains a section, which we call $\sigma_\infty \cong \mathcal{V}_x$, and at most a finite number of further points. Let U_x be the inverse image of $\mathcal{V}_x$ in the universal family.

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Since all curves are immersed at x, the tangent morphism of i_x gives a nowhere vanishing morphism of vector bundles,

$$T\, i_x : T_{U_x|\mathcal{V}_x|\sigma_\infty} \to i_x^*(T_{X|x}).$$

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Assume, by contradiction, that Φ_x is not finite: by the above morphism, we can find a curve $C \subset \mathcal{V}_x$ such that N_{σ_∞, U_x} is trivial along C.

But σ_∞ is contracted and the normal bundle must be negative.

The Tangent Map

The next result was proved in general by Hwang and Mok.

Theorem

For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, the tangent morphism $\Phi_x : \mathcal{V}_x \longrightarrow P(T_xX)$ is birational (i.e. generically injective) over its image.

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Therefore Φ_x is the normalization of its image in $P(T_xX)$.

The Tangent Map if tangent bundle is ample

Note (c.f Mori '79 Corollary 7.ii) that if TX is ample
(in particular $-K_X$ is ample and X is uniruled)
and we take a locally unsplit (unbreakable) family of rational curves, $\mathcal{V}$,
then for **every** element $[f] \in \mathcal{V}$ we have

$$f^*TX = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

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Thus it is an etale cover of $P(T_xX) = \mathbb{P}^{n-1}$. But $\mathbb{P}^{n-1}$ is simply connected and therefore Φ_x is birational and thus an isomorphism.

The Variety of Minimal Rational Tangents

Definition

We define $S_x \subset P(T_x X)$ as the closure of the image of the map Φ_x and we call it *tangent cone of curves from $\mathcal{V}$ at the point x.*

J.-M.Hwang and N. Mok call this *Variety of Minimal Rational Tangents.* The name tangent cone follows from the fact that S_x is (at least around $[f]$) the tangent cone to $\mathrm{Locus}(V_x)$.

Variety of Minimal Rational Tangents

For our purposes we need the following observation which follows from the above discussion.

Lemma

The projectivised tangent space of the tangent cone S_x at $\Phi_x([f])$ is equal to $P((f^*TX)_0^+) \subset P((f^*TX)_0) = P(T_xX)$.

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Proof The tangent space to $\mathrm{Locus}(V_x)$ at $f(p)$, for $p \neq 0$, is the image of the evaluation of sections of the twisted pull-back of TX which is

$$\mathrm{Im}(T\hat{F})_p = (f^*TX)_p^+ \subset (f^*TX)_p = T_{f(p)}X.$$

Thus passing with p to 0 we get the result.

Characterization of $\mathbb{P}^n$

The following is the celebrated Theorem of Mori of 1979.

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Let X be a complex projective manifold of dimension $n \geq 3$. Assume that TX is ample. Then X is isomorphic to the projective space.

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The next Theorem was first proved by Cho-Miyaoka-Shepherd Barron; subsequently Kebekus gave a shorter proof.

Theorem

Let X be a complex projective manifold of dimension $n \geq 3$. Assume that for every curve $C \subset X$ we have $-K_X \cdot C \geq n+1$. Then X is isomorphic to the projective space.

Note that Mori's Theorem follows immediately from it.

Proof

Proof. Take an unbreakable uniruling $\mathcal{V}$. By our assumption and the above results, for a general point $x \in X$ we have that $\mathcal{V}_x$ is smooth and $dim(\mathcal{V}_x) = -K_X \cdot C - 2 = (n-1)$.

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By the above results we have that $\mathcal{V}_x \cong \sigma_\infty \cong \mathbb{P}^{n-1}$.

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Let $\tilde{i}_x : \mathcal{V}_x \to \tilde{X} = Bl_x X$ be the lift up of i_x; since Ti_x has rank one along σ_∞, then $T\tilde{i}_x$ has maximal rank along σ_∞, in particular $N_{\sigma_\infty, U_x} \cong N_{E/\tilde{X}} = O_{\mathbb{P}^{n-1}}(-1)$.

Proof

Consider the Stein factorization of the universal map $i_x : U_x \to X : U_x \to Y \to X$, where the first map $\alpha : U_x \to Y$ contracts the divisor σ_∞ and the second $\beta : Y \to X$ is a finite map.

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Since $R^1\pi_*(\mathcal{O}_{U_x}) = 0$ and $\mathcal{O}_{U_x}(\sigma_\infty)_{|\sigma_\infty} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, the push forward of the twisted ideal sheaf sequence

$$0 \to \mathcal{O}_{U_x} \to \mathcal{O}_{U_x}(\sigma_\infty) \to \mathcal{O}_{U_x}(\sigma_\infty)_{|\sigma_\infty} \to 0$$

gives on $\mathcal{V}_x \cong \mathbb{P}^{n-1}$ a sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^{n-1}} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to 0,$$

where $U_x \cong \mathbb{P}(\mathcal{E}^*)$. Since $Ext^1_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1), \mathcal{O}_{\mathbb{P}^{n-1}}) = 0$, then $U_x \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$.

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An application of Zariski's main theorem shows that α is the standard contraction of σ_∞, that is $Y = \mathbb{P}^n$.

Proof

We have that adiunction formula for a finite, surjective morphism:

$$-K_{\mathbb{P}^n} = \beta^*(-K_X) + \text{branch divisor.}$$

Let l be a line through $\alpha(x)$ and $t = \beta(l)$; t is a curve associated with $\mathcal{V}_x$.

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Thus we have
$n+1 = -K_X \cdot t = (\beta^*(-K_X)) \cdot l = (-K_{\mathbb{P}^n} - (\text{branch divisor})) \cdot l =$
$n+1 - (\text{branch divisor}) \cdot l$

Then the branch divisor is empty and β is birational, thus an isomorphism.

Another generalization

The following generalization of Mori's is due to A. and Wisniewski.

Theorem

Let X be a complex projective manifold of dimension $n \geq 3$. Assume that there exist a subsheaf $E \subset TX$ which is an ample vector bundle. Then X is isomorphic to the projective space.

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Proof. By the assumption we can apply the Theorem of Miyaoka, therefore X is uniruled.

Take an unbreakable uniruling $\mathcal{V}$: for a general $f \in \mathcal{V}$ we have $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus d} \oplus \mathcal{O}^{\oplus(n-d-1)}$,where $d = deg(f^*(-K_X)) - 2$.

Proof

Lemma

For any $f \in \mathcal{V}$ the pull-back f^*E is isomorphic either to $\mathcal{O}(1)^{\oplus r}$ or to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus (r-1)}$. In particular the family of curves parametrized by V is unsplit.

Proof. For a general $f \in \mathcal{V}$ the pull-back f^*E is an ample subbundle of $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus (d)} \oplus \mathcal{O}^{\oplus (n-d-1)}$ and thus it is as in the lemma.

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Since $deg(f^*E) = r$ or $deg(f^*E) = r + 1$ and $r > 1$, and for any ample bundle $\mathcal{E}$ over a rational curve we have $deg(\mathcal{E}) \geq rank(\mathcal{E})$, it follows that no curve from V can be split into a sum of two or more rational curves, hence V is unsplit.

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Lemma

Let X, E and $\mathcal{V}$ be as above and moreover assume that $\varphi^0 : X^0 \to Z^0$ is an rc$\mathcal{V}$ fibration. Then E is tangent to a general fiber of φ^0.
That is, if X_g is a general fiber of φ^0, then the injection $E_{|X_g} \to TX_{|X_g}$ factors via $E_{|X_g} \hookrightarrow TX_g$.

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This implies that $E_{|X_g} \to TX_{|X_g}$ factors to $E_{|X_g} \to TX_g$ generically and since the map $TX_g \to TX_{|X_g}$ has cokernel which is torsion free (it is the normal sheaf which is locally free) this yields $E_{|X_g} \hookrightarrow TX_g$, a sheaf injection.

Proof

Proposition

The general fiber of φ^0, X_g, is $\mathbb{P}^k$ and $E_{|X_g} = \mathcal{O}(1)^{\oplus r}$ or $E_{|X_g} = TX_g$.

Proof By abuse we denote the general fiber with $X := X_g$. We consider here only the case when for $f \in \mathcal{V}$ the pull-back f^*E is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$. In particular $f^*E \subset (f^*TX)^+$.

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Comparing the splitting type of f^*E and f^*TX we see that the tangent map $Tf : T\mathbb{P}^1 \to f^*TX$ factors to a vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \to f^*E$. (In other words, we have surjective morphism $(f^*E)^* \to \Omega_{\mathbb{P}^1} \cong \mathcal{O}(-2)$).

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The general fiber of φ^0, X_g, is $\mathbb{P}^k$ and $E_{|X_g} = \mathcal{O}(1)^{\oplus r}$ or $E_{|X_g} = TX_g$.

Proof By abuse we denote the general fiber with $X := X_g$. We consider here only the case when for $f \in \mathcal{V}$ the pull-back f^*E is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$. In particular $f^*E \subset (f^*TX)^+$.

Comparing the splitting type of f^*E and f^*TX we see that the tangent map $Tf : T\mathbb{P}^1 \to f^*TX$ factors to a vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \to f^*E$. (In other words, we have surjective morphism $(f^*E)^* \to \Omega_{\mathbb{P}^1} \cong \mathcal{O}(-2)$).

The vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \to f^*E$ implies $(f^*TX)^+ \hookrightarrow f^*E$. In fact, choose a general f which is an immersion at $0 \to x$. Then $\Phi_x([f]) \in P(E_x) = P((f^*E)_0) \subset P(T_xX) = P((f^*TX)_0)$ and the same holds for morphisms in a neighborhood of $[f]$ in V_x. Thus around $\Phi_x([f])$ the tangent cone S_x is contained in $P(E_x) = P((f^*E)_0)$, so is its tangent space $P((f^*TX)_0^+)$.

Proof

Therefore $f^*E = (f^*TX)^+$ and thus $\deg(f^*E) = \deg(f^*(-K_X))$.
Since $\rho(X) = 1$ it follows that $\det(E) = -K_X$.

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The embedding $E \hookrightarrow TX$ gives rise to a non-trivial morphism $\det(E) \to \Lambda^r TX$ and thus to a non-zero section of $\Lambda^r TX \otimes K_X$. We use dualities to have the equalities:

$$h^0(X, \Lambda^r TX \otimes K_X) = h^n(X, \Omega_X^r) = h^r(X, \Omega_X^n) = h^r(X, K_X) = h^{n-r}(X, \mathcal{O}_X)$$

and, since X is Fano, the latter number is non-zero only if $r = n$.

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We conclude by the Theorem of Mori.

Proof

Finally we prove that $dimZ_0$ is zero , i.e. X is rationally connected.
By contradiction if $dimZ_0 \geq 1$ one can prove that :

Lemma

Outside a subset of codimension ≥ 2 the morphism φ_0 is a $\mathbb{P}^k$-bundle (in the analytic topology).

Then we take a complete curve $B \subset Z_0$ and we consider the $\mathbb{P}^k$-bundle $\varphi_0 : X_B := \varphi_0^{-1}(B) \to B$ with the ample vector bundle $E_{|X_B}$.

Proof

We get a contradiction applying the following straightforward result, due to Campana and Peternell.

Lemma

Let X be a n-dimensional projective manifold, $\varphi : X \to Y$ a $\mathbb{P}^k$ bundle $(k < n)$ of the form $X = \mathbb{P}(V)$ with a vector bundle V on Y. Then the relative tangent sheaf $T_{X/Y}$ does not contain an ample locally free subsheaf