Rational Curves on Symplectic Manifolds

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A **symplectic variety** is a normal variety Y whose smooth part admits a holomorphic symplectic form ω_Y such that its pull back to any resolution $\pi : X \to Y$ extends to a holomorphic 2-form ω_X on X.
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More generally, a map $\pi : X \to Y$ is called a **symplectic contraction** if X is a symplectic manifold, Y is normal and π is a birational projective morphism. If moreover Y is affine we will call $\pi : X \to Y$ a **local symplectic contraction** or **local symplectic resolution**.

Properties of Symplectic Varieties

Proposition

Let Y be a symplectic variety and $\pi : X \to Y$ be a resolution. Then the following statement are equivalent:
- (i) $\pi^* K_Y = K_X$,
- (ii) π is symplectic,
- (iii) K_X is trivial,
- (iv) for every symplectic form on Y_{reg} its pull-back extends to a symplectic form on X.
Note that Y is Gorenstein and K_Y is trivial.

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Corollary

By the Grauert Riemeschneider Teorem

$$0 = R^i \pi_* K_X = R^i \pi_* \mathcal{O}_X$$

for all positve i. In particular
- Y has rational singularities.
- All exceptional fibers of π are uniruled.

Theorem

Let $\pi : X \to Y$ be a symplectic resolution with $dim X = 2n$. Let also $f : \mathbb{P}^1 \to X$ be a non constant morphism such that $f(\mathbb{P}^1)$ is a π-exceptional curve. Then

$$dim\ Hom_f(\mathbb{P}^1, X) \geq 2n + 1.$$

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Theorem

Let X be a compact symplectic manifold. Then the deformation space (the Kuranishi space) of the complex structure of X is smooth and its tangent space at $[X]$ is excactly $H^1(X, \Omega^1_X)$. Moreover, given a homology class $\alpha \in H_2(X, \mathbb{Q}) = H^{2,0}(X)^*$ represented by a rational 1-cycle, there is a one-parameter deformation $\mathcal{X} = \{X_t\}_{t \in T}$ such that the flat lifting $\alpha \in H_2(X_t, \mathbb{Q})$ of α is no more an algebraic cycle for general $t \in T$.

Rational Curves

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$$dim_{[f]}Hom(\mathbb{P}^1, \chi) = \chi(\mathbb{P}^1, f^*T_\chi) \geq dim\chi - degf^*K_\chi \geq 2n + 1.$$

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Since all the deformation of f stays in X then $dim\ Hom_f(\mathbb{P}^1, X) = dim\ Hom_g(\mathbb{P}^1, \chi)$ and we are done.

Semismallness

As a corollary we have the following Theorem (**semismall property**).

Theorem

A symplectic resolution $\pi : X \to Y$ is semismall, that is for every closed subvariety $Z \subset X$ we have $2\operatorname{codim} Z \geq \operatorname{codim} \pi(Z)$. If equality holds Z then is called a maximal cycle.

Semismallness

Sketch of proof: let $F \subset X$ be a generic fiber of $Z \to \pi(Z)$, let also $d = dimZ$ and $e = dim(\pi(Z))$.
We know that all exceptional fibers are uniruled; take then $\mathcal{V}$ be a generically unsplit family which covers F and let $V := u^{-1}(\mathcal{V})$.

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By previous disequalities we have $dimV = dimLocus(V) + dimLocus(V, 0 \to x) + 1 \leq 2dimF + 1 = 2d - 2e + 1.$

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Let $f : \mathbb{P}^1 \longrightarrow F$ be a rational curve in V; since $f(\mathbb{P}^1)$ gets contracted under π, all its deformations in X stay in the exceptional set and we may assume that all small deformations stay in Z.

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Therefore $dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, X) = dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, Z) = dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, F) + e \leq 2d - e + 1$.

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By the above Theorem we have on the other hand that $dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, X) \geq 2n + 1$ and the Theorem follows.

Local symplectic contractions in dimension 4

Let $\pi : X \to Y$ be a local symplectic contraction, $\dim X = 4$.
By the semismall property, the fibers of π have dimension less or equal to 2. We will denote with 0 the unique (up to shrinking Y to a smaller affine set) point such that $\dim \pi^{-1}(0) = 2$. If π is divisorial then the general non trivial fiber has dimension 1.

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The following theorem is a sort of *relative* characterization of the projective space: the hard part is to prove that the two dimensional fiber is normal.
(Proved by Wierzba-Wisniewski and in any dimension by Cho-Miyaoka-Shepherd-Barron.)

Theorem

Suppose that π is small (i.e. it does not contract any divisor). Then π is locally analytically isomorphic to the collapsing of the zero section in the cotangent bundle of $\mathbb{P}^2$.
Therefore X admits a Mukai flop

4-dimensional local symplectic contr. are MDS

The above theorem, together with Matsuki's termination of 4-dimensional flops, is the key ingredient in the proof of the following result.

Theorem

Let $\pi : X \to Y$ be a 4-dimensional local symplectic contraction and let $\pi^{-1}(0)$ be its only 2-dimensional fiber. Then X is a Mori Dream Space over Y. Moreover any SQM model of X over Y is smooth and any two of them are connected by a finite sequence of Mukai flops whose centers are over $0 \in Y$. In particular, there are only finitely many non isomorphic (local) symplectic resolutions of Y.

Examples: quotient symplectic singularities

Let S be a smooth surface and denote by $S^{(n)}$ the *symmetric product* of S, that is $S^{(n)} = S^n/\sigma_n$, where σ_n is the group of permutations.
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Suppose now that $S \to S'$ is a resolution of a Du Val singularity: $S' = \mathbb{C}^2/H$ with $H < SL(2, \mathbb{C})$ a finite group. The composition

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Note that $(S')^{(n)}$ is a quotient singularity with respect to the action of the wreath product $H \wr \sigma_n = (H^n) \rtimes \sigma_n$ (σ_n permutes factors in $H^n = H^{\times n}$).

Central fiber in resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$

Let consider the case $n = 2$ and $H := \mathbb{Z}_3 < SL(2)$.
Note that $\mathbb{Z}_3 \wr \mathbb{Z}_2$ has another nice presentation, namely $D_6 \rtimes \mathbb{Z}_3$, where D_6 is the dihedral group and $\mathbb{Z}_3$ acts on it by rotations.

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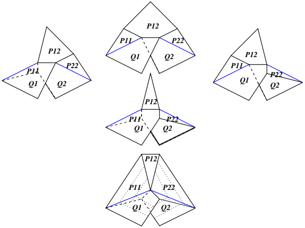
The Figure presents a description of configurations of components in the special fiber of different symplectic resolutions



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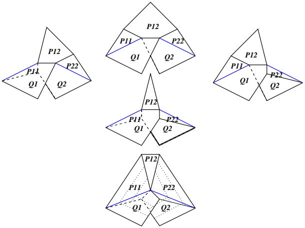

The configuration at the top is the Hilb-Chow map.
The one in the bottom, obtained by first resolving the singularities of the action of $D_6 = \sigma_3$ and then by resolving the singularities of the $\mathbb{Z}_3$ action, is called a $D_6 \rtimes \mathbb{Z}_3$-resolution.

The Chow Scheme in dimension 4

Let $\pi : X \to Y$ is a local symplectic divisorial contraction, $\dim X = 4$, $D \subset X$ the exceptional locus, $dimD = 3$, and $S = \pi(D) \subset Y$ ($dimS = 2$). A general fiber of π over any component of S is a configuration of $\mathbb{P}^1$'s with dual graph being a Dynkin diagram (Wierzba).

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Choose $S' \subset S$ an irreducible component, $C \cong \mathbb{P}^1$ irreducible curve in a (general) fiber over a point in $S' \setminus \{0\}$ and let D' be the irreducible component of D which contains C.
Let $\mathcal{V} \subset Chow(X/Y)$ be (the normalization of) an irreducible component of the Chow scheme of X containing C, $p : \mathcal{U} \to \mathcal{V}$ be the universal family and $q : \mathcal{U} \to D' \subset X$ be the evaluation map.

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The contraction π determines a morphism $\tilde{\pi} : \mathcal{V} \to S'$, let $\mu : \mathcal{V} \to \widetilde{S'} \to S'$ be its Stein factorization.
The exceptional locus of μ is $\mu^{-1}(\nu^{-1}(0)) = \bigcup_i V_i$ where $V_i \subset \mathcal{V}$ are irreducible curves.

The Chow Scheme in dimension 4

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "q"] \arrow[d, "p"'] & & D' \subset X \arrow[d, "\pi"'] \\ \mathcal{V} \arrow[r, "\mu"] & \widetilde{S}' \arrow[r, "\nu"] & S' \subset Y \end{tikzcd} \tag{4.0.1}$$

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Theorem

The surface $\widetilde{S}'$ has at most Du Val singularity at $\nu^{-1}(0)$ and $\mu : \mathcal{V} \to \widetilde{S}'$ is its, possibly non-minimal, resolution. In particular every V_i is a rational curve.

The differential

Consider the following maps:
- derivative map $Dq : q^*\Omega_X \to \Omega_{\mathcal{U}}$
- $Dp : p^*\Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}} \longrightarrow \Omega_{\mathcal{U}/\mathcal{V}} \longrightarrow 0$,
- its dual $0 \longrightarrow T_{\mathcal{U}/\mathcal{V}} \longrightarrow T_{\mathcal{U}} \longrightarrow p^*T_{\mathcal{V}}$.
- the isomorphism $\omega_X : T_X \to \Omega_X$ given by the symplectic form ω_X on X.

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- the isomorphism $\omega_X : T_X \to \Omega_X$ given by the symplectic form ω_X on X.

Collect them in the following diagram

$$\begin{tikzcd}
T_{\mathcal{U}/\mathcal{V}} \arrow[r] & T_{\mathcal{U}} \arrow[r, "(Dp)^*"] \arrow[d, "(Dq)^*"] & p^*(T_{\mathcal{V}}) & p^*(\Omega_{\mathcal{V}}) \arrow[d, "Dp"] & \\
& q^*T_X \arrow[r, "q^*(\omega_X)"'] & q^*\Omega_X \arrow[r, "Dq"] & \Omega_{\mathcal{U}} \arrow[r] & \Omega_{\mathcal{U}/\mathcal{V}}
\end{tikzcd} \tag{4.0.2}$$

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Claim: the dotted arrow exists and it is obtained by a pull back of a two form $\omega_{\mathcal{V}}$ on $\mathcal{V}$.

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Dq is of maximal rank outside of $p^{-1}(\bigcup_i V_i)$ and p is just a $\mathbb{P}^1$-bundle there. Thus $\omega_{\mathcal{V}}$ does not assume zero outside the exceptional set of μ. Hence $K_{\mathcal{V}} = \sum a_i V_i$, with $a_i \geq 0$ being the discrepancy of V_i.

Chow scheme in resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$

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In the previous example we denote by $\mathcal{V}_0$ the component of $Chow(X/Y)$ dominating S' and parametrizing curves equivalent to e_0; it will change under flops.

Lemma

If X is the Hilbert-Chow resolution then $\mathcal{V}_0$ is the minimal resolution of A_2 singularity. If X is the $D_6 \rtimes \mathbb{Z}_3$-resolution then $\mathcal{V}_0$ is non-minimal, with one (-1) curve in the central position of three exceptional curves.

Thank you for your attention

Busan lighthouse