

## Lifting

Marco Andreatt

Weighted Blow-up

Lifting Weightee Blow-up

 $\Delta^{r}$ -MMP

Extremal rays

Divisorial contractions to a point

Conjectures and Results

## Lifting from an ample section

## Marco Andreatta

Dipartimento di Matematica di Trento, Italia

Milano, 2019

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# Weighted Projective Space

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Conjectures and Results Let  $\sigma = (a_1, \ldots, a_n) \in \mathbb{N}^n$  such that  $a_i > 0$  and  $gcd(a_1, \ldots, a_n) = 1$ ; let  $M = lcm(a_1, \ldots, a_n)$ .



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The weighted projective space with weight  $(a_1, \ldots, a_n)$ , denoted by  $\mathbb{P}(a_1, \ldots, a_n)$ , can be defined either as:

$$\mathbb{P}(a_1,\ldots,a_n):=(\mathbb{C}^n-\{0\})/\mathbb{C}^*,$$

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where  $\xi \in \mathbb{C}^*$  acts by  $\xi(x_1, ..., x_n) = (\xi^{a_1}x_1, ..., \xi^{a_n}x_n)$ .



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where  $\xi \in \mathbb{C}^*$  acts by  $\xi(x_1, ..., x_n) = (\xi^{a_1}x_1, ..., \xi^{a_n}x_n)$ . Or as:

$$\mathbb{P}(a_1,\ldots,a_n):=\operatorname{Proj}_{\mathbb{C}}\mathbb{C}[x_1,\ldots,x_n],$$

where  $\mathbb{C}[x_1, ..., x_n]$  is the polynomial algebra over  $\mathbb{C}$  graded by the condition  $deg(x_i) = a_i$ , for i = 1, ..., n.



# **Cyclic quotient singularities**

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is an affine variety definite as the quotient of  $\mathbb{C}^n$  by the action  $\epsilon : (x_1, ..., x_n) \to (\epsilon^{a_1} x_1, ..., \epsilon^{a_n} x_n)$ , where  $\epsilon$  is a primitive *m*-th root of unity.



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Equivalently *X* is isomorphic to the spectrum of the ring of invariant monomials under the group action,

$$X = Spec \ \mathbb{C}[x_1, ..., x_n]^{\mathbb{Z}_m}$$

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Equivalently *X* is isomorphic to the spectrum of the ring of invariant monomials under the group action,

$$X = Spec \ \mathbb{C}[x_1, ..., x_n]^{\mathbb{Z}_m}$$

Let  $Q \in Y : (g = 0) \subset \mathbb{C}^{n+1}$  be a hypersurface singularity with a  $\mathbb{Z}_m$  action. The point  $P \in Y/\mathbb{Z}_m := X$  is called a *hyperquotient singularity*.

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given by  $(x_1,\ldots,x_n)\mapsto (x_1:\ldots:x_n)$ .



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Conjectures and Results Let  $X = \mathbb{C}^n / \mathbb{Z}_m(a_1, ..., a_n)$  be a cyclic quotient singularity and consider the rational map

$$\varphi: X \to \mathbb{P}(a_1, \ldots, a_n)$$

given by  $(x_1,\ldots,x_n)\mapsto (x_1:\ldots:x_n)$ .

## Definition

The weighted blow-up of  $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, ..., a_n)$  with weight  $\sigma = (a_1, ..., a_n)$  (or simply the  $\sigma$ -blow-up),  $\overline{X}$ , is defined as the closure in  $X \times \mathbb{P}(a_1, ..., a_k)$  of the graph of  $\varphi$ , together with the morphism  $\pi_{\sigma} : \overline{X} \to X$  given by the projection on the first factor.

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Namely, let  $e_i = (0, ..., 1, ..., 0)$  for i = 1, ..., n and  $e = 1/m(a_1, ..., a_n)$ . *X* is the toric variety which corresponds to the lattice  $\mathbb{Z}e_1 + ... + \mathbb{Z}e_n + \mathbb{Z}e$  and the cone  $C(X) = \mathbb{Q}_+e_1 + ... + \mathbb{Q}_+e_n$  in  $\mathbb{Q}^n$ .

 $\pi_{\sigma}: \overline{X} \to X$  is the proper birational morphism from the normal toric variety  $\overline{X}$  corresponding to the cone decomposition of C(X) consisting of  $C_i = \sum_{j \neq i} \mathbb{Q}_+ e_j + \mathbb{Q}_+ e$ , for i = 1, ..., n, and their intersections.



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defined on a monomial  $M = x_1^{s_1} \dots x_n^{s_n}$  as  $\sigma$ -wt $(M) := \sum_{i=1}^n s_i a_i / m$ .



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defined on a monomial  $M = x_1^{s_1} \dots x_n^{s_n}$  as  $\sigma$ -wt $(M) := \sum_{i=1}^n s_i a_i / m$ . For any  $d \in \mathbb{N}$  we define the  $\sigma$ -weighted ideal of degree d as

$$I_{\sigma,d} = \{g \in \mathbb{C}[x_1,\ldots,x_n] : \sigma\text{-wt}(g) \ge d\} = (x_1^{s_1}\cdots x_n^{s_n} : \sum_{j=1}^n s_j a_j \ge d).$$

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## Proposition

The weighted blow-up of of  $X = \mathbb{C}^n / \mathbb{Z}_m(a_1, ..., a_n)$  with weight  $\sigma$ ,  $\pi : \overline{X} \to X$ , is given by

$$\overline{X} = \operatorname{Proj}_X(\bigoplus_{d \ge 0} I_{\sigma,d}) \to X.$$



Definition

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# Let $X : ((g = 0) \subset \mathbb{C}^{n+1})/\mathbb{Z}_m(a_0, ..., a_n)$ be a hyperquotient singularity and let $\pi : \overline{\mathbb{C}^{n+1}}/\mathbb{Z}_m(a_0, ..., a_n) \to \mathbb{C}^{n+1}/\mathbb{Z}_m(a_0, ..., a_n)$ be the $\sigma = (a_0, ..., a_n)$ -blow-up.

Let  $\overline{X}$  be the proper transform of X via  $\pi$  and call again, by abuse,  $\pi$  its restriction to  $\overline{X}$ .

Then  $\pi : \overline{X} \to X$  is also called the weighted blow-up of X with weight  $\sigma = (a_1, ..., a_n)$  (or simply the  $\sigma$ -blow-up).

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restriction to  $\overline{X}$ . Then  $= \sqrt{\overline{X}}$   $\rightarrow$  X is else called the weighted blow up of X with weight

Then  $\pi : X \to X$  is also called the *weighted blow-up* of X with weight  $\sigma = (a_1, ..., a_n)$  (or simply the  $\sigma$ -blow-up).

## Proposition

Let  $X : ((g = 0) \subset \mathbb{C}^{n+1})/\mathbb{Z}_m(a_0, ..., a_n)$  be a hyperquotient singularity and let  $i : X \to \mathbb{C}^{n+1}/\mathbb{Z}_m(a_0, ..., a_n)$  be the inclusion. Then

$$\overline{X} = \mathbb{P}_X igl( \mathcal{O}_X \oplus igoplus_{d \in \mathbb{N}, d > 0} J^\sigma(db) igr) o X,$$

where  $J^{\sigma}(db) := i^{-1} (I^{\sigma}(db))^{\cdot} \mathcal{O}_X$ .



# **3-dimensional Terminal Singularities**

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In dimension 3 they consist of hyperquotient singularities whose associated hypersurfaces in  $\mathbb{C}^4$  could be finetely listed.



# **Fano-Mori contractions**

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This was further carried on by M. Kawakita, T.Hayakawa and J. A. Chen.



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They are all weighted blow-ups of (particular) cyclic quotient or hyperquotient singularities and this should be the case for the few remaining ones. The following is probably a Theorem:

## Conjecture

The divisorial contractions to a point for a MMP in dimension 3 are weighted blow-up of a specific list of hyperquotient singularities.

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Conjectures and Results The crucial technical result of our approach:

## Theorem

Let  $f : X \to Z$  be a local, projective,  $\mathbb{Q}$ -factorial contraction, which contracts an irreducible divisor E to an isolated  $\mathbb{Q}$ -factorial singularity  $P \in Z$ . Assume that dimX > 4.

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Let  $Y \subset X$  be a *f*-ample Cartier divisor such that  $f' = f_{1Y} : Y \to f(Y) = W$  is a  $\sigma' = (a_1, \dots, a_{n-1})$ -blow-up.



# Lifting WBU

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Divisorial contractions to a point

Conjectures and Results The crucial technical result of our approach:

## Theorem

Let  $f : X \to Z$  be a local, projective,  $\mathbb{Q}$ -factorial contraction, which contracts an irreducible divisor E to an isolated  $\mathbb{Q}$ -factorial singularity  $P \in Z$ .

Assume that  $dim X \ge 4$ .

Let  $Y \subset X$  be a *f*-ample Cartier divisor such that  $f' = f_{|Y} : Y \to f(Y) = W$  is a  $\sigma' = (a_1, \dots, a_{n-1})$ -blow-up.

Then  $f: X \to Z$  is  $a \sigma = (a_1, \ldots, a_{n-1}, a_n)$ -blow-up,  $\pi_{\sigma}: X \to Z$ , where  $a_n$  is such that  $Y \sim_f -a_n E$  ( $\sim_f$  means linearly equivalent over f).

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Conjectures and Results The Theorem could be seen as a generalization of the first paper by S. Mori: *On a generalization of complete intersections*, J. Math. Kyoto Univ., 1975.



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Since then new spaces were born, in particular the so called Mori Dream Spaces: their Cox Rings are finitely generated.



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The Cox Rings of weighted blow-ups determines completely the blow-up; this is true for any Toric variety.



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## Question

Does the Cox Ring of an ample section determine the one of the variety?

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# I part: Lifting CQS

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Conjectures and Results

## Proposition

Let Z be an affine variety of dimension  $n \ge 4$  with an isolated  $\mathbb{Q}$ -factorial singularity at  $P \in Z$ .

Assume that  $(W, P) \subset (Z, P)$  (germs of complex spaces around P) is a Weil divisor which is a cyclic quotient singularity, i.e.  $W = \mathbb{C}^{n-1}/\mathbb{Z}_m(a_1, ..., a_{n-1}).$ 

Then Z is a cyclic quotient singularity, i.e.  $Z = \mathbb{C}^n / \mathbb{Z}_m(a_1, ..., a_{n-1}, a_n)$ .



# **Proof of the proposition**

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Conjectures and Results If *W* is a Cartier divisor, i.e. *W* is given as a zero locus of a regular function *f*,  $W : (f = 0) \subset Z$ , consider the map  $f : Z \to \mathbb{C}$ . It is is flat, since dim<sub> $\mathbb{C}$ </sub> $\mathbb{C} = 1$ .

Quotient singularities of dimension bigger or equal then three are rigid, by a fundamental theorem of M. Schlessinger. Since Z has an isolated singularity and  $dimW = n - 1 \ge 3$ , it implies that W is smooth, i.e. m = 1. A variety containing a smooth Cartier divisor is smooth along it,

m = 1. A variety containing a smooth Cartier divisor is smooth along it, therefore, eventually shrinking around *P*, *Z* is also smooth.

In the general case we use Reid's trick: since Z is Q-factorial, we can assume that there exists a minimal positive integer r such that rW is Cartier (r is the index of W). Take a Galois cover  $\pi : Z' \to Z$ , with group  $\mathbb{Z}_r$ , such that Z' is normal,  $\pi$  is etale over  $Z \setminus P$ ,  $\pi^{-1}(P) =: Q$  is a single point and the Q-divisor  $\pi^*W := W'$  is Cartier,  $W' : (f' = 0) \subset Z'$ .



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Conjectures and Results Let *b* a positive integer such that  $-bE \sim Y$  (and therefore Cartier). By Grothendieck theory  $X = \operatorname{Proj}_{\mathcal{O}_Z}(\bigoplus_{d \geq 0} f_*(\mathcal{O}_X(-dbE)))$ , therefore we want to prove that

$$f_*(\mathcal{O}_X(-dbE) = I_{\sigma,db} = (x_1^{s_1} \cdots x_n^{s_n} \mid \sum_{j=1}^n a_j s_j \ge db).$$



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Conjectures and Results Consider the exact sequence on *X* 

$$0 \to \mathcal{O}_X(-Y - dbE) \to \mathcal{O}_X(-dbE) \to \mathcal{O}_Y(-dbE) \to 0$$

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ightarrow \mathcal{O}_X(-Y-dbE) 
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ightarrow 0$$

Pushing it down via  $\varphi$  and using the relative Vanishing theorems we have

$$0 \to f_*\mathcal{O}_X(-(d-1)bE) \xrightarrow{\cdot x_n} f_*\mathcal{O}_X(-dbE) \to f_*\mathcal{O}_Y(-dbE) \to 0.$$

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The proposition follows by induction on *n* 

$$(f_*(\mathcal{O}_Y(-dbE) = (x_1^{s_1} \cdots x_{(n-1)}^{s_{(n-1)}} \mid \sum_{j=1}^{n-1} a_j s_j \ge db)))$$

and on d

$$(f_*(\mathcal{O}_X(-(d-1)bE) = (x_1^{s_1} \cdots x_n^{s_n} \mid \sum_{j=1}^n a_j s_j \ge (d-1)b))$$



# Counterexample in dimension 3, I part

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## Remark

If n = 3 we have the following example.

## Example

Let  $Z' = \mathbb{C}^4/\mathbb{Z}_r(a, -a, 1, 0)$ ; let (x, y, z, t) be coordinates in  $\mathbb{C}^4$  and assume (a, r) = 1. Let  $Z \subset Z'$  be the hypersurface given as the zero set of the function  $f := xy + z^{rm} + t^n$ , with  $m \ge 1$  and  $n \ge 2$ . This is a terminal singularity which is not a cyclic quotient (it is a terminal hyperquotient singularity).

However the surface  $W := Z \cap (t = 0)$ , which is the surface in  $\mathbb{C}^3/\mathbb{Z}_r(a, -a, 1)$  given as the zero set of  $(xy + z^{rm})$ , is a cyclic quotient singularity of the type  $\mathbb{C}^2/\mathbb{Z}_{r^2m}(a, rm - a)$ .



# **Counterexample in dimension 3,**

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## Example

Let  $X = \mathbb{P}^2 \times \mathbb{P}^1$ 

*Z* be a  $\mathbb{F}_1$  surface in the linear system  $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1, 1)$ ;

the contraction of the (-1) curve of Z lifts to the  $\mathbb{P}^1$ -bundle contraction onto  $\mathbb{P}^2$ .



# **Minimal Model Program- BCHM**

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Conjectures and Results Consider a log pair  $(X, \Delta)$ , i.e a normal variety *X* and an effective  $\mathbb{R}$  divisor  $\Delta$ , which is Kawamata log terminal (klt) (that is  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier and for a (any) log resolution  $g : Y \to X$  we have  $g^*(K_X + \Delta) = K_Y + \Sigma b_i \Gamma_i$  with  $b_i < 1$ , for all *i*).

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If  $\Delta$  is big by BCHM on a klt log pair  $(X, \Delta)$  we can run a  $K_X + \Delta$ - Minimal Model Program with scaling:  $(X_0, \Delta_0) = (X, \Delta) \rightarrow (X_1, \Delta_1) \rightarrow - - - \rightarrow (X_s, \Delta_s)$ 



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$$(X_0, \Delta_0) = (X, \Delta) \to (X_1, \Delta_1) \to --- \to (X_s, \Delta_s)$$
  
such that:



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1)  $(X_i, \Delta_i)$  is a klt log pair, for i = 0, ..., s;



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Conjectures and Results

$$(X_0, \Delta_0) = (X, \Delta) \rightarrow (X_1, \Delta_1) \rightarrow --- \rightarrow (X_s, \Delta_s)$$
  
such that:

1)  $(X_i, \Delta_i)$  is a klt log pair, for i = 0, ..., s;

2)  $\varphi_i : X_i \to X_{i+1}$  is a birational map which is either a divisorial contraction or a flip associated with an extremal ray  $R_i = \mathbb{R}^+[C_i]$  such that  $(K_{X_i} + \Delta_i) \cdot C_i < 0$ (notation:  $R_i \in \overline{NE(X_i)}_{(K_{X_i} + \Delta_i) < 0} \subset \overline{NE(X_i)}_{K_{X_i} < 0}$ )



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3) either  $K_{X_s} + \Delta_s$  is nef (i.e.  $(X_s, \Delta_s)$  is a log Minimal Model), or  $X_s \to Z$  is a Mori fiber space relatively to  $K_{X_s} + \Delta_s$ (depending on the pseudeffectivity of  $K_X + \Delta$ ).



# MMP for a q.p. pair- Adjunction Theory

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Conjectures and Results Let (X, L) be a quasi-polarized variety and let  $r \in \mathbb{Q}^+$ . Lemma (zip L into a boundary). Since *L* is nef and big there exists an effective  $\mathbb{Q}$ -divisor  $\Delta^r$  on *X* such that

 $rL \sim_{\mathbb{Q}} \Delta^r$  and  $(X, \Delta^r)$  is Kawamata log terminal.



# MMP for a q.p. pair- Adjunction Theory

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 $rL \sim_{\mathbb{Q}} \Delta^r$  and  $(X, \Delta^r)$  is Kawamata log terminal.

Run a  $K_X + \Delta^r$ -MMP and get a birational klt pair  $(X_s, \Delta_s^r)$  which is - either a Minimal Model  $(K_{X_s} + \Delta_s \text{ is nef})$ 

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- or  $X_s \to Z$  is a Mori fiber space relatively to  $K_{X_s} + \Delta_s$ .



# MMP for a q.p. pair- Adjunction Theory

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- or  $X_s \to Z$  is a Mori fiber space relatively to  $K_{X_s} + \Delta_s$ .

## Remarks/Problems

- $(X_s, \Delta_s^r)$  is not necessarily an (r) q.p. pair, i.e. we do not have a priori a nef and big Cartier divisor  $L_s$  such that  $rL_s \sim_{\mathbb{Q}} \Delta_s^r$ .
- Beyond the existence of the MMP, it would be nice to have a "description" of each steps and eventually of the Mori fiber spaces.



## **Extremal rays**

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Divisorial contractions to a point

Conjectures and Results For the above program we study the (Fano-Mori) contractions:

 $\varphi:X\to Y$ 

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associated to a rays  $R = \mathbb{R}^+[C] \in \overline{NE(X)}_{(K_X+rL)<0} \subset \overline{NE(X)}_{K_X<0}$ .



## **Extremal rays**

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That is  $\varphi$  is a projective map between normal variety, with connected fibers, X has terminal  $\mathbb{Q}$ -factorial singularities and an irreducible curve  $C \subset X$  is mapped to a point by  $\varphi$  iff  $[C] \in R$ .

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## **Extremal rays**

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 $\varphi$  can be of fiber type (dimX > dimY), a Mori fiber space, or birational, more precisely either divisorial or small.



# **Apollonio method**

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Conjectures and Results Let  $X' \in |L|$  a generic divisor with "good singularities".



# **Apollonio method**

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#### Extremal rays

Divisorial contractions to point

Conjectures and Results Let  $X' \in |L|$  a generic divisor with "good singularities".

-  $\varphi_{|X'} := \varphi' : X' \to Y'$  is the Fano-Mori contraction associated to  $R' \in \overline{NE(X')}_{(K_{X'}+(r-1)L')<0}$ .



# **Apollonio method**

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Conjectures and Results Let  $X' \in |L|$  a generic divisor with "good singularities".

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- Any section of L on X' extends to a section of L on X.



# **Base point free technique**

is of fiber type.

Lifting	
Lifting Weighted Blow-up	
Extremal rays	Theorem (Fano, Fujita, Kawamata, Kollar, Shokurov,,
Divisorial contractions to a point Conjectures and Results	A-Wisniewski, Mella, A-Tasin)
	If dim $F \le r + 2$ then there exists $X' \in  L $ with "good" singularities (i.e. as in X), except for two cases in which $n = 3$ , dim $F = r + 2$ and $\varphi$

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#### Lifting

#### Marco Andreatt

Weighted Blow-up

Theorem

- Lifting Weighted Blow-up
- $\Delta^r$ -MMP
- Extremal rays

#### Divisorial contractions to a point

Conjectures and Results Let  $\varphi : X \to Y$  be a birational contraction in a  $K_X + \Delta^{n-2}$ -MMP (i.e. it is associated with an extremal ray on a q.p. pair such that  $R = \mathbb{R}^+[C] \in \overline{NE(X)}_{(K_X+(n-2)L)<0} \subset \overline{NE(X)}_{K_X<0}$  and  $L \cdot C > 0$ ).



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Then  $\varphi : X \to Y$  is the weighted blow-up of a smooth point in Y of weights (1, 1, b, ..., b), where b is a natural positive number.



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#### Lifting

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### Definition

Theorem

We call such  $\varphi$  a Castelnuovo-Kawakita contraction.



## Proof

#### Lifting

Marco Andreatta

Weighted Blow-up

Lifting Weighted Blow-up

 $\Delta^r$ -MMP

Extremal rays

Divisorial contractions to a point

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By the base point free theorem, we can assume *L* is very ample.



## Proof

#### Lifting

Marco Andreatta

Weighted Blow-up

Lifting Weighted Blow-up

 $\Delta^r$ -MMP

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Conjectures and Results We have dim F > (n - 2); thus dim F = (n - 1) and  $\varphi$  is a contraction of a divisor to a point.

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Thus we get the existence of sections in |L| with terminal singularities. Inductively, slicing with (n-2) general sections of |L|, we can reduce to the case of a Fano Mori contraction on a surface,  $f' : S \to W$ . Surfaces with terminal singularities are smooth. Apply now **Castelnuovo's Theorem** to have that W is smooth and f' is a (1, 1)-blow-up.



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Apply the "lifting of weighted blow-up".



# Divisorial contractions in the (n-3)-MMP

#### Lifting

Marco Andreatt

Weighted Blow-up

Lifting Weighted Blow-up

 $\Delta^r$ -MMP

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Conjectures and Results

### Theorem

Let X be a variety with  $\mathbb{Q}$ -factorial terminal singularities of dimension  $n \ge 3$  and let  $f : X \to Z$  be a local, projective, divisorial contraction which contracts a prime divisor E to an isolated  $\mathbb{Q}$ -factorial singularity  $P \in Z$  such that  $-(K_X + (n - 3)L)$  is f-ample, for a f-ample Cartier divisor L on X.

*Then*  $P \in Z$  *is a hyperquotient singularity and f is a weighted blow-up.* 

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Conjectures and Results

## **Conjecture** (Mori)

Let X be a projective variety with canonical singularities, if  $k(X) = -\infty$  then X is uniruled

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It is false for general singularities, for instance for  $\mathbb{Q}$ -Gorenstein rational.

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Following Mori's approach, BDPP and then BCHM proved, using the bend and breaking theory, the following

### Theorem

Let X be a projective variety with canonical singularities, if  $K_X$  is not pseudoeffective then X is uniruled



#### Lifting

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Equivalently  $K_X$  pseudo-effective  $\implies \exists N \in \mathbb{N}^+ \text{ s.t. } H^0(X, NK_X) \neq \emptyset$ 

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Equivalently  $K_X$  pseudo-effective  $\implies \exists N \in \mathbb{N}^+$  s.t.  $H^0(X, NK_X) \neq \emptyset$ 

The following result is Theorem D in BCHM

### Theorem (BCHM)

Let (X, L) be a quasi polarized pair and t > 0.



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The following result is Theorem D in BCHM

### Theorem (BCHM)

Let (X, L) be a quasi polarized pair and t > 0.  $K_X + tL$  pseudo-effective  $\Longrightarrow \exists N \in \mathbb{N}^+$  s.t.  $H^0(X, N(K_X + tL)) \neq \emptyset$ 

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# Effectivity of non vanishing

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### Theorem (BCHM)

Let (X, L) be a quasi polarized pair and t > 0.

 $K_X + tL \text{ pseudo-effective} \Longrightarrow \exists N \in \mathbb{N}^+ \text{ s.t. } H^0(X, N(K_X + tL)) \neq \emptyset$ 

The next Conjecture is an effective version of the above Theorem.

### Conjecture

Let (X, L) be a quasi polarized pair and t > 0.  $K_X + tL$  pseudo-effective  $\Longrightarrow H^0(X, (K_X + tL)) \neq \emptyset$ 

For t = 1 this is a version of the so-called Ambro-Ionescu-Kawamata conjecture, which is true for  $n \le 3$ . For t = n - 1 this is a conjecture by Beltrametti and Sommese.



# **Some Results**

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Conjectures and Results A proof of the Conjecture should go through a  $K_X + \Delta^t$ -MMP with  $tL \sim_{\mathbb{Q}} \Delta^t$  and  $(X, \Delta^t)$  is Kawamata log terminal, as explained above.



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The Conjecture is true for t = n. This case happens if and only if the pair (X, L) is birationally equivalent (via a 0-reduction) to the pair  $(\mathbb{P}^n, \mathcal{O}(1))$ .



# **Some Results**

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For t = n - 1 the conjecture was essentially proved by Höring (if *L* is effective). We prove a slightly more explicit version of his result, namely, we show that this case happens if and only if the pair (X, L) is birationally equivalent (via a 1-reduction) to a finite list of pairs.


## **Some Results**

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If t = n - 2 and n = 4 the conjecture is true (Fukuma and some generalizations).



# **Termination of Adjunction**

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## Conjecture

Let (X, L) be a quasi polarized pair. If  $H^0(X, (mK_X + L)) = \emptyset$  for all  $m \ge m_o > 0$  then X is uniruled.

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# **Termination of Adjunction**

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Let (X, L) be a quasi polarized pair. If  $H^0(X, (mK_X + L)) = \emptyset$  for all  $m \ge m_o > 0$  then X is uniruled.

This Conjecture is true for surfaces (Castelnuovo and Enriques) and for threefolds.

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It follows from the above Conjecture (for arbitrary small t > 0).