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# FAMILIES OF RATIONAL CURVES WHICH DETERMINE THE STRUCTURE OF THE (PROJECTIVE) SPACE

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ABSTRACT. We work on algebraic uniruled varieties: we define special families of rational curves, we introduce the tangent map and we use it to characterize some special varieties. Finally we extend the above technique to the case of complex symplectic (projective) manifold.

## 1. HILBERT SCHEME

Let $X$ and $Y$ be a normal projective schemes (of finite type over $k$ algebraically closed field).

We denote with $\mathrm{Hilb}(X)$ the Hilbert scheme of proper subschemes of $X$; with $\mathrm{Hom}(Y, X)$ the open subscheme of $\mathrm{Hilb}(X \times Y)$ of morphisms from $Y$ to $X$ (the construction of the schemes is due to Grothendieck and Mumford).

**Theorem 1.1.** *Let $f : Y \longrightarrow X$ be a morphism. Assume that $Y$ is without embedded points and that $X$ has no embedded points contained in $f(Y)$ and the image of every irreducible component of $Y$ intersect the smooth locus of $X$. Then*

- *The tangent space of $Hom(Y, X)$ at $[f]$ is naturally isomorphic to*
$$Hom_Y(f^*{\Omega_X}^1, \mathcal{O}_Y).$$
- *The dimension of every irreducible component of $Hom(Y, X)$ at $[f]$ is at least*
$$dim Hom_Y(f^*{\Omega_X}^1, \mathcal{O}_Y) - dim {Ext_Y}^1(f^*{\Omega_X}^1, \mathcal{O}_Y).$$

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Let $f : C \longrightarrow X$ be a morphism from a proper curve to a scheme and $L$ a line bundle on $X$. We use the following notation to denote the intersection number of $C$ and $L$:

$$C \cdot L := deg_C f^* L$$

In the special case of the Hilbert scheme of curves, thank to Riemann Roch theorem, we have the following nice result.

**Theorem 1.2.** *Let $C$ be a proper algebraic curve without embedded points and $f : C \longrightarrow X$ a morphism to a smooth variety $X$ of pure dimension $n$. Then*

$$dim_{[f]} Hom(C, X) \geq -K_X \cdot C + n\chi(\mathcal{O}_C).$$

*Moreover equality holds if $H^1(C, f^*T_X) = 0$.*

*Proof.* If $F$ is a locally free sheaf on a scheme $Z$, then $\mathrm{Ext}^i_X(F, \mathcal{O}_Z) = H^i(Z, F^*)$. Therefore we have $dim_{[f]}\mathrm{Hom}(C, X) \geq dim\mathrm{Hom}_C(f^*{\Omega_X}^1, \mathcal{O}_C) - dim\mathrm{Ext}_C{}^1(f^*{\Omega_X}^1, \mathcal{O}_C) = h^0(C, f^*T_X) - h^1(C, f^*T_X) = \chi(C, f^*T_X) = deg f^*T_X + n\chi(\mathcal{O}_C) = -K_X \cdot C + n\chi(\mathcal{O}_C)$.

*Remark* 1.3. Let $f : \mathbb{P}^1 \to X \in Hom(\mathbb{P}^1, X)$ and assume $X$ is smooth along $f(\mathbb{P}^1)$. Then the tangent space of $\mathrm{Hom}(\mathbb{P}^1, X)$ at $[f]$ is naturally isomorphic to $\mathrm{Hom}_Y(f^*{\Omega_X}^1, \mathcal{O}_Y) = H^0(\mathbb{P}^1, f^*T_X)$.

## 2. Existence of Rational Curves

A *rational curve* on $X$ is a non constant morphism $\mathbb{P}^1 \longrightarrow X$.
The following is a fundamental result of S. Mori ([Mo82]).

**Theorem 2.1.** *Let $X$ be a smooth projective variety over an algebraically closed field (of any characteristic), $C$ a smooth, projective and irreducible curve and $f : C \longrightarrow X$ a morphism. Assume that*

$$-K_X \cdot C > 0.$$

*Then for every $x \in f(C)$ there is a rational curve $D_x \subset X$ containing $x$ and such for any nef $\mathbb{R}$-divisor $L$ :*

$$L \cdot D_x \leq 2dimX\left(\frac{L \cdot C}{-K_X \cdot C}\right) \quad \textit{and} \quad -K_X \cdot D_x \leq dimX + 1.$$

**Idea of Proof.** If $C$ has genus 0, then we are done. Let $g := g(C) > 0$ and $n = dimX$.

Step 1. We have seen that

$$dim_{[f]}\mathrm{Hom}(C,Y) \geq -K_Y \cdot C + n(1-g).$$

Take $x = f(0) \in f(C)$; since $n$ conditions are required to fix the image of the basepoint 0 under $f$, morphisms $f$ of $C$ into $X$ sending 0 to $x$ have a deformation space of dimension

$$\geq -K_Y \cdot C + n(1-g) - n = -K_Y \cdot C - ng.$$

Thus whenever this quantity is positive there must be a non-trivial one-parameter family of deformations of the map $f$ keeping the image of 0 fixed.

In particular, we can find a nonsingular (affine) curve $D$ and a morphism $g : C \times D \to X$, thought of as a nonconstant family of maps, *all sending* 0 *to the same point* $x$.

Step 2. We argue now that $D$ cannot be complete (Bend and Break). In fact otherwise consider $U$, a neighborhood of $x$ in $C$ and the projection map $\pi : U \times D \to U$.

Then $\pi$ is a proper, surjective morphism with connected fiber of dimension 1. Moreover $g(\pi^{-1}(x))$ is a single point.

By the rigidity Lemma, $g(\pi^{-1}(y)$ is a single point for all $y$ in $U$, i.e. the family would have to be constant.

**Proof of the rigidity Lemma:** (see [KM98] Lemma 1.6)

Let $W = im(\pi \times g) \subset U \times X$ and consider the proper morphisms

$$\pi : U \times D \to W \to U,$$

where the first map $h = U \times D \to W$ is defined by $h(t) = (\pi(t), g(t))$ and the second $p$ is the projection to the first factor.

$p^{-1}(y) = h(\pi^{-1}(y))$ and dim $p^{-1}(x) = 0$; by the upper semicontinuity of fiber dimension there is an open set $x \in V \subset U$ such that dim $p^{-1}(y) = 0$ for every $y \in V$. Thus $h$ has fiber dimension 1 over $p^{-1}(V)$, hence $h$ has fiber dimension at least 1 everywhere.

For any $w \in W$, $h^{-1}(w) \subset \pi^{-1}(p(w)$, dim $h^{-1}(w) \geq 1$ and dim $\pi^{-1}(p(w)) = 1$. Therefore $h^{-1}(w)$ is a union of irreducible components of $\pi^{-1}(p(w)$, and so $h(\pi^{-1}(p(w))) = p^{-1}(p(w))$ is finite. It is a single point since $\pi^{-1}(p(w))$ is connected.

Step 3. So let $D \subset \overline{D}$ be a completion where $\overline{D}$ is a nonsingular projective curve. Let $G : C \times \overline{D} \dashrightarrow X$ be the rational map defined by $g$. Blow up a finite number of points to resolve the undefined points to get $Y \to C \times \overline{D}$ whose composition given by $\pi : Y \to X$ is a morphism. Let $E \subset Y$ be the exceptional curve of the last blow up. Since it was actually needed, it can't be collapsed to a point, and hence $\pi(E)$ is our desired curve.

Step 4. If $char(k) = p > 0$ we consider another curve $h : C \to C \subset X$, where $h$ is a composition of $f$ with a $r$ power of the Frobenius endomorphism $F_p$.
Roughly speaking if the curve $C$ is given as zero set of algebraic equations in the variable $(y_0, ..., y_m)$, then $F_p : (y_0, ..., y_m) \to (y_0^p, ..., y_m^p)$. $F_p : C \to C$ is injective set-theoretically but it is an endomorphism of degree $p$. Take $h = F_p^r \circ f$ and call $C'$ and $C''$ the curves respectively given as image of $f$ and of $h$.

We only change the structure sheaf and not the topological space, so both curve has genus $g$.

But $-C''\cdot K_X = -p^r(C'\cdot K_X)$ and for $r$ high enough we have $-C''\cdot K_X \geq ng + 1$. Therefore

$$-K_Y \cdot C'' - ng > 0.$$

In this way we prove the existence of a rational curve through $x$ for almost all $p > 0$.

Step 5. Algebra.
Principle. If a homogeneous system of algebraic equations with integral coefficients has a non trivial solution in $\overline{\mathbb{F}}_p$ for infinitely many $p$, then it has a non trivial solution in any algebraically closed field.

A map $\mathbb{P}^1 \to X \subset \mathbb{P}^N$ of limited degree with respect to $-K_X$ can be given by a system of equations. Since this system has a non trivial solution for infinitely many $p$, it has a solution in any algebraically closed field by the above principle.

**Definition 2.2.** A normal proper variety is called *uniruled* if it is covered by rational curves.

The above Theorem proves that Fano manifolds are uniruled.

The following Theorem of Y. Miyaoka, which generalizes the Mori's result, is the most powerful uniruledness criteria.

**Theorem 2.3.** *Let $X$ be a smooth and proper variety over $\mathbb{C}$. Then $X$ is uniruled if and only if there is a quotient sheaf ${\Omega_X}^1 \longrightarrow F$ and a family of curves $\{C_t\}$ covering an open subset of $X$ such that $F_{|C_t}$ is locally free and $deg(F_{|C_t}) < 0$ for every $t$.*

## 3. Families of rational curves

**Definition 3.1.** Let $\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the open suscheme corresponding to those morphisms $f : \mathbb{P}^1 \longrightarrow X$ which are birational onto their image, that is $f$ is an immersion at its generic point. This is an open condition.
If $f : \mathbb{P}^1 \longrightarrow X$ is any morphism and $h \in Aut(\mathbb{P}^1)$, then$f \circ h$ is "counted" as a different morphism. The group $Aut(\mathbb{P}^1)$ acts on $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ and it is the quotient that "really parametrizes" morphisms of $\mathbb{P}^1$ into $X$.
It can be proved that the quotient exists in the sense of Mumford (Mori-Mumford-Fogarty) ; its normalization will be denoted$RatCurves^n(X)$ and called the *space of rational curve on* $X$.

Given a point $x \in X$, one can similarly find a scheme $\mathrm{Hom}(\mathbb{P}^1, X, [0:1] \to x)$ whose geometric points correspond to generically injective morphisms from $\mathbb{P}^1$ to $X$ which map the point $[0:1]$ to $x$. The quotient, in the sense of Mumford, by the group of automorphism of $\mathbb{P}^1$ which fixes the point $[0:1]$, will be denoted by $RatCurves^n(x, X)$ and called the *space of rational curves through* $x$.

**Definition 3.2.** An irreducible component $\mathcal{V}$ of $RatCurves^n(X)$ is called a *family of rational curve*.

We obtain a diagram as follows

(3.0.1)
$$\begin{tikzcd} \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \arrow[r, "U"] \arrow[d] & Univ(X) \arrow[r, "i"] \arrow[d, "\pi"'] & X \\ \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \arrow[r, "u"] & RatCurves^n(X) & \end{tikzcd}$$

where $U$ and $u$ have the structure of principal $Aut(\mathbb{P}^1)$-bundle; $\pi$ is a $\mathbb{P}^1$-bundle. The restriction of $i$ to any fiber of $\pi$ is generically injective, i.e. birational onto its image.

Similarly for a given point $x \in X$:

(3.0.2)
$$\begin{tikzcd} \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, [0:1] \to x) \times \mathbb{P}^1 \arrow[r, "U"] \arrow[d] & Univ(x, X) \arrow[r, "i_x"] \arrow[d, "\pi"'] & X \\ \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, [0:1] \to x) \arrow[r, "u"] & RatCurves^n(x, X) & \end{tikzcd}$$

Let $B = \emptyset$ or $x$.

Let $F : \mathbb{P}^1 \times Hom^n_{bir}(\mathbb{P}^1, X, B) \to X$ be the *universal map* defined by $F(f, p) = f(p)$; $F$ is the composition $i_B \circ U$.

Let $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$, be an irreducible component and $V_x$ be the set of elements in $V$ passing through $x \in X$.

We denote $Locus(V) := F(\mathbb{P}^1 \times V)$ and $Locus(V, 0 \to x) := F(\mathbb{P}^1 \times V_x)$.

Let $f : \mathbb{P}^1 \to X \in Hom^n_{bir}(\mathbb{P}^1, X, B)$ and assume $X$ is smooth along $f(\mathbb{P}^1)$. Then the tangent space of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, B)$ at $[f]$ is naturally isomorphic to

$$T_{[f]}\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, B) = \mathrm{Hom}_Y(f^*\Omega^1_X(-B), \mathcal{O}_Y) = H^0(\mathbb{P}^1, f^*T_X(-B)).$$

In particular the tangent map of $F$ at the point $(f, t)$ :

(3.0.3)
$$dF_{f,t} : H^0(\mathbb{P}^1, f^*T_X(-B)) \oplus T_{\mathbb{P}^1,t} \to T_{X,f(t)}$$

is given by

$$(\sigma, u) \to (df_t(u) + \sigma(t)).$$

## 4. Special families of rational curves

We know that any vector bundle over $\mathbb{P}^1$ splits as a direct sum of line bundles, this is sometime called a Grothendieck' s Theorem.

Let $X$ be a smooth projective variety of dimension $n$ and $B = \emptyset$ (respectively $B = \{x\}$). Let $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$, be an irreducible component; for a rational curve $f : \mathbb{P}^1 \longrightarrow X$ in $V$ we therefore have

$$f^*TX \otimes I_B = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n).$$

The splitting type, i.e. the $a_i$, are the same for all members $f \in V$.

By the general theory of Hilbert scheme (i.e. by Theorem 1.2 and its proof) we easily see that if $a_i \geq -1$ (resp. $a_i \geq 0$), then $V$ (resp. $V_x$) is generically smooth and $dimV = dimX + \Sigma a_i$.

Moreover, since $dimLocus(V) = rk(dF)$ at a generic point $x \in X$, using the description of the tangent map of $F$ (see 3.0.3), we have :

$$dimLocus(V) = \sharp\{i : a_i \geq 0\};$$

similarly , for general $x \in X$:

$$dimLocus(V_x) = \sharp\{i : a_i \geq 1\}.$$

**Definition 4.1.** $f$ is called **free** (over $B$) if $a_i \geq 0$ for every $i$. Equivalently $f$ is free if $f^*TX \otimes I_B$ is generated by its global sections and $H^1(\mathbb{P}^1, f^*T(X) \otimes I_B) = 0$ (for more see [K095] II.3).

From the above observations we have immediately the following.

**Proposition 4.2.** *(Assume $char(k) = 0$). $X$ is uniruled if and only through a general point $x \in X$ there is a free rational curve.*

*Remark* 4.3. Note that $X$ is uniruled if and only if there exists a family of rational curve $\mathcal{V}$ such that $i : Univ(X) \longrightarrow X$ is dominant. (This follows from the fact that the irreducible components of $RatCurves^n(X)$ are numerable; which in turn follows from the fact that families of a given degree, with respect to a very ample line bundle, are finite, depending on the Hilbert polynomial).
In this case we call $\mathcal{V}$ a **unruling** for $X$.

**Theorem 4.4.** *Let $\mathcal{V}$ be an irreducible component of $RatCurves^n(X)$. Denote by $\mathcal{V}^{free} \subset \mathcal{V}$ the parameter space of members of $\mathcal{V}$ that are free. Then $\mathcal{V}$ is a uniruling if and only if $\mathcal{V}^{free}$ is nonempty. In this case, $\mathcal{V}^{free}$ is a Zariski open subset of the smooth locus of $\mathcal{V}$.*

Given a uniruling $\mathcal{V}$ on $X$ and a point $x \in X$, let $\mathcal{V}_x$ be the normalization of the subvariety of $\mathcal{V}$ parametrizing members of $\mathcal{V}$ passing through $x$. Since by the above Theorem non-free rational curves do not cover $X$, for general point $x \in X$, the structure of $\mathcal{V}_x$ is particularly nice ([K095] II.3.11):

**Theorem 4.5.** *For a uniruling $\mathcal{V}$ on a projective manifold $X$ and a general point $x \in X$, all members of $\mathcal{V}_x$ belongs to $\mathcal{V}^{free}$. Furthermore, the variety $\mathcal{V}_x$ is a finite union of smooth quasi-projective varieties of dimension $deg_{K_X^{-1}}(\mathcal{V}) - 2$.*

**Definition 4.6.** A family of rational curve $\mathcal{V}$ on a projective manifold $X$ is **locally unsplit** or **unbreakable** if $\mathcal{V}_x$ is projective for a general $x \in X$. Members of an unbreakable uniruling on $X$ will be called minimal rational curves on $X$. (In [K095] IV.2.1 the definition is given for all points $x \in X$, not only general, and it is called unsplit family).

Unbreakable unirulings exist on any uniruled projective manifold. To see this, we need the following notion.

**Definition 4.7.** Let $L$ be an ample line bundle on a projective manifold $X$. A uniruling $\mathcal{V}$ is **minimal** with respect to $L$, if $deg_L(\mathcal{V})$ is minimal among all unirulings of $X$. A uniruling is a minimal uniruling if it is minimal with respect to some ample line bundle.

**Proposition 4.8.** *Minimal unirulings exist on any uniruled projective manifold. A minimal uniruling is unbreakable. In particular there exist unbreakable unirulings on any uniruled projective manifold.*

Sketchy geometric proof: suppose for a uniruling $\mathcal{V}$, which is minimal with respect to an ample line bundle $L$, the variety $\mathcal{V}_x$ is not projective for a general point $x \in X$. Then the members of $\mathcal{V}_x$ degenerate to reducible curves all components of which are rational curves of smaller $L$-degree than the members of $\mathcal{V}$ and some components of which pass through $x$. Collecting those components passing through $x$ as $x$ varies over the general points of $X$ gives rise to another uniruling $\mathcal{V}'$ satisfying $deg_L(\mathcal{V}') < deg_L(\mathcal{V})$, a contradiction to the minimality of $deg_L(\mathcal{V})$.

Mori shows that a weaker version of this property continues to hold for unbreakable unirulings:

**Theorem 4.9.** *Let $\mathcal{V}$ be an unbreakable family. Then for a general point $x \in Locus(\mathcal{V})$ and any other point $y \in Locus(\mathcal{V}_x)$, there does not exist a positive-dimensional family of members of $\mathcal{V}$ that pass through both $x$ and $y$.*
*(This is the definition of generically unsplit family in* [K095] *I.V.2.1. The Theorem is* [K095] *I.V.Proposition 2.3.)*

The theorem is proved again by a "bend-and-break" plus rigidity argument. Geometrically, it says that any 1-dimensional family of rational curves which share two distinct points in common must degenerate into a reducible curve. This is the most important geometric property of an unbreakable uniruling.

If $\mathcal{V}$ is unbreakable and we let $V = u^{-1}(\mathcal{V})$ and $\Pi : V \to X \times X$ be the map $[f] \to (f(0), f(\infty))$, the Theorem 4.9 says that the fiber of $\Pi$ over the generic point of Im $(\Pi)$ has dimension at most one.

Using this formulation of the Theorem we have the following:

**Proposition 4.10.** *Let $\mathcal{V}$ be an unbreakable family and let $V = u^{-1}(\mathcal{V})$.*
*If $x \in X$ is a general point in $Locus(V)$, then*

$$dimV = dimLocus(V) + dimLocus(V, 0 \to x) + 1.$$

*Proof.* The proof follows from the semi continuity of the fiber dimension ([K095] IV.2.5).
By upper-semicontinuity, for $x \in Locus(V)$

$$dim\{[f] \in V : f(0) = x\} \geq dimV - dimLocus(V).$$

If $y \in Locus(V_x)$ similarly

$$dim\{[f] \in V : f(0) = x, f(\infty) = y\} \geq dimV - dimLocus(V) - dim(Locus(V_x),$$

equality holds for general $x$ and $y$.
The proposition follows since

$$1 = dim\Pi^{-1}(x, y) = dim\{[f] \in V : f(0) = x, f(\infty) = y\}.$$

□

Combining this Theorem with 1.2 we obtain the following result.

**Corollary 4.11.** *Let $\mathcal{V}$ be an unbreakable family and let $V := u^{-1}(\mathcal{V})$. Then*

- $dimX + deg_{-K}V \leq dimLocus(V) + dimLocus(V, 0 \to x) + 1$
- $dimX + deg_{-K}V \leq 2dimLocus(V) + 1 \leq 2dimX + 1$
- $deg_{-K}V \leq dimLocus(V, 0 \to x) + 1 \leq dimX + 1$

**Definition 4.12.** A rational curve $C \subset X$ is **unbending** if under the normalization $v_C : \mathbb{P}^1 \longrightarrow C \subset X$, the vector bundle ${v_C}^* T(X)$ has the form

$${v_C}^* T(X) = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$$

for some integer $p$ satisfying $0 \leq p \leq n-1$, where $n = dimX$.
(This is the definition of Minimal free morphism in [K095] I.V.2.8.)

The above definition allows an infinitesimal version of Theorem 4.9.

**Theorem 4.13.** *A general member of an unbreakable uniruling is unbending.*

*Proof.* (See [K095] 2.9, 2.10) Sketch of proof: Let $[f] \in V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ a general element of an irreducible component $V$ which is an unbreakable uniruling (i.e. $V = u^{-1}\mathcal{V}$ with $\mathcal{V}$ an unbreakable uniruling).
Let $f^*TX = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$. By assumption $a_i \geq 0$ for every $i$. Then

$dimX + \Sigma a_i = dimV = dimLocus(V) + dimLocus(V, 0 \to x) + 1 = dimX + \sharp\{i : a_i \geq 1\} + 1.$

Therefore $\Sigma a_i = \sharp\{i : a_i \geq 1\} + 1$, that is at most one of the $a_i$ is at least two.

*Remark* 4.14. If $f : \mathbb{P}^1 \to C \subset X$ is an unbending member of $\mathcal{V}_x$ the differential $Tf : T(\mathbb{P}^1) \longrightarrow f^*T(X)$ is an isomorphism of $T(\mathbb{P}^1)$ and the unique $\mathcal{O}(2)$ summand. Therefore $Tf_p$ is non zero at every $p \in \mathbb{P}^1$. Recall that a curve is *immersed* if its normalization has rank one at every point; therefore an unbending member is immersed.

**Definition 4.15.** A family of rational curve $\mathcal{V}$ on a projective manifold $X$ is **unsplit** if $\mathcal{V}$ is projective.

Let $\mathcal{V}$ be an unsplit uniruling. It defines a relation of *rational connectedness with respect to* $\mathcal{V}$, which we shall call rc$\mathcal{V}$ relation for short, in the following way: $x_1,\ x_2 \in X$ are in the rc$\mathcal{V}$ relation if there exists a chain of rational curves parametrized by morphisms from $\mathcal{V}$ which joins $x_1$ and $x_2$. The rc$\mathcal{V}$ relation is an equivalence relation and its equivalence classes can be parametrized generically by an algebraic set. More precisely, we have the following result due to Campana and, independently, to Kollár-Miyaoka-Mori.

**Theorem 4.16.** *(see* [K095]*, IV.4.16). There exist an open subset $X^0 \subset X$ and a proper surjective morphism with connected fibers $\varphi^0 : X^0 \to Z^0$ onto a normal variety, such that the fibers of $\varphi^0$ are equivalence classes of the rc$\mathcal{V}$ relation.*

We shall call the morphism $\varphi^0$ an rc$\mathcal{V}$ fibration. If $Z_0$ is just a point then we will call $X$ a rationally connected manifold with the respect to the family $\mathcal{V}$, in short an rc$\mathcal{V}$ manifold.

**Lemma 4.17.** *Let $X$ be a manifold which is rationally connected with the respect to a unsplit uniruling $\mathcal{V}$. Then $\rho(X) := dim N_1(X) = 1$ and $X$ is a Fano manifold.*

Also in this case the proof is a sort of an (easy) bend and break lemma.

## 5. Tangent Map

Let $X$ be a smooth projective variety and $\mathcal{V} \subset RatCurves^n(X)$, a closed irreducible component; fix a point $x \in X$ and consider $\mathcal{V}_x$.

**Definition 5.1.** The rational map $\Phi_x : \mathcal{V}_x ----> P(T_xX)$, defined, at $[f] \in \mathcal{V}_x$ which is smooth at 0 , by $\Phi_x([f]) = [(Tf)_0(\partial/\partial t)]$, c.f. [Mori79, pp.602-603], is called the tangent map.
It sends a member of $\mathcal{V}_x$ which is smooth at 0 to its tangent direction.

By $P$ we denote the "natural projectivisation" (that is vector spaces modulo homotheties) in opposition to "Grothendieck projectivisation" (that is projective spectrum of the symmetric algebra of a vector space) which we denote by $\mathbb{P}$. With $t$ we denote a local coordinate around $0 \in \mathbb{P}^1$.

**Proposition 5.2.** *If $f : \mathbb{P}^1 \to C \subset X$ is an unbending member of $\mathcal{V}_x$, the tangent map above defined can be extended to $[f]$, even when $C$ is singular at $x$, because the differential $Tf : T(\mathbb{P}^1) \longrightarrow f^*T(X)$ is injective.*
*Moreover $\Phi_x$ is immersive at $[f] \in \mathcal{V}_x$.*

*Proof.* It remains to prove that $\Phi_x$ is immersive; we take it from [Hw01] Proposition 1.4. Let $V = u^{-1}\mathcal{V}$ the Hilbert family corresponding to $\mathcal{V}$; we have seen that $T_{[f]}\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, B) = H^0(\mathbb{P}^1, f^*T_X(-B))$, $B = \emptyset$ or $x$. Passing to the quotient by $Aut(\mathbb{P}^1)$, i.e. passing to $\mathcal{V}$, we delete the part corresponding to $T(\mathbb{P}^1)$; since ${v_C}^*T(X) = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$, we have

$$T_{[f]}(\mathcal{V}_x) = \oplus\mathcal{O}^p \oplus \mathcal{O}(-1)^{n-1-p} \subset T_{[f]}(\mathcal{V}) = \oplus\mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}.$$

Take $v \in T_{[f]}(\mathcal{V}_x) subset T_{[f]}(\mathcal{V})$; we can find a deformation $f_t$ of $f_0 := f$ such that $\frac{df}{dt}_{|t=0} = v$. Let $z$ be a local coordinate in $\mathbb{P}^1$ centered at 0.
Then the differential

$$d\Phi_x : T_{[f]}(\mathcal{V}_x) \to T_{\Phi_x([f])}P(T_xX)$$

send $v$ to $d\Phi_x(v) = \frac{d}{dt}_{|t=0}\frac{df_t}{dz}_{|z=0} = \frac{d}{dz}_{|z=0}\frac{df_t}{dt}_{|z=0} = \frac{dv}{dz}_{|z=0}$.
To derive $v$ with respect to $z$ we think it in $T_{[f]}(\mathcal{V}) = \oplus\mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$; a non zero section here has non vanishing differential. □

In particular, Theorem 4.13 implies that for an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, the tangent map $\Phi_x$ is generically finite over its image.

Kebekus has carried out an analysis of singularities of members of $\mathcal{V}_x$ and proved that they are considerably well behaved. Among other things, he ([Keb02] [Theorem 3.3] has shown

**Theorem 5.3.** *For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, members of $\mathcal{V}_x$ which are singular are a finite number. Moreover the singular ones are immersed at the point corresponding to $x$.*

Using this, Kebekus has shown the following ([Keb02-2] [Theorem 3.4].

**Theorem 5.4.** *For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$ (as in Theorem 5.3), the tangent morphism $\Phi_x : \mathcal{V}_x \longrightarrow P(T_xX)$ can be defined by assigning to each member $C$ of $\mathcal{V}_x$ its tangent direction.*
*This morphism $\Phi_x$ is finite over its image.*

*Proof.* Let $i_x$ be the map in 3.0.2; by 5.3 the preimage $i_x^{-1}(x)$ contains a section, which we call $\sigma_\infty \simeq \mathcal{V}_x$, and at most a finite number of further points. Let $U_x$ be the inverse image of $\mathcal{V}_x$ in the universal family.
Since all curves are immersed at $x$, the tangent morphism of $i_x$ gives a nowhere vanishing morphism of vector bundles,

$$Ti_x : T_{U_x|\mathcal{V}_x|\sigma_\infty} \to i_x^*(T_{X|x}).$$

The tangent map $\Phi_x$ is then given by the projectivization of the above map. Assuming that $\Phi_x$ is not finite, then we can find a curve $C \subset \mathcal{V}_x$ such that $N_{\sigma_\infty, U_x}$ is trivial along $C$. But $\sigma_\infty$ can be contracted and the normal bundle must be negative.

The next result was proved in a special case (when the tangent map is surjective) by Kebekus, in general it has been proved by Hwang and Mok [HM04].

**Theorem 5.5.** *For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$ (as in Theorem 5.3), the tangent morphism $\Phi_x : \mathcal{V}_x \longrightarrow P(T_xX)$ is birational (i.e. generically injective) over its image.*

Theorems 5.4 and 5.5 says that $\Phi_x$ is the normalization of its image in $P(T_xX)$.

**Definition 5.6.** We define $S_x \subset P(T_xX)$ as the closure of the image of the map $\Phi_x$ and we call it *tangent cone of curves from $\mathcal{V}$ at the point $x$.*

J.-M.Hwang and N. Mok call this *Variety of Minimal Rational Tangents*. The name tangent cone follows from the fact that $S_x$ is (at least around $[f]$) the tangent cone to $\mathrm{Locus}(V_x)$.

For our purposes we need the following observation which follows from the above discussion (sse also Proposition 2.3 in [Hw01]).

**Lemma 5.7.** *The projectivised tangent space of the tangent cone $S_x$ at $\Phi_x([f])$ is equal to $P((f^*TX)_0^+) \subset P((f^*TX)_0) = P(T_xX)$.*

*Proof.* By 3.0.3 the tangent space to $\mathrm{Locus}(V_x)$ at $f(p)$ for $p \neq 0$ is the image of the evaluation of sections of the twisted pull-back of $TX$ which is $\mathrm{Im}(T\hat{F})_p = (f^*TX)_p^+ \subset (f^*TX)_p = T_{f(p)}X$. Thus passing with $p$ to 0 we get the result.

## 6. Characterization of $\mathbb{P}^n$

The following is the celebrated Theorem of Mori of 1979 ([Mo79].

**Theorem 6.1.** *Let $X$ be a complex projective manifold of dimension $n \geq 3$. Assume that $TX$ is ample. Then $X$ is isomorphic to the projective space.*

The next Theorem was first proved by Cho-Miyaoka-Shepherd Barron; subsequently Kebekus gave a shorter proof in [Keb02-2].
Note that Mori's Theorem follows immediately from it.

**Theorem 6.2.** *Let $X$ be a complex projective manifold of dimension $n \geq 3$. Assume that for every curve $C \subset X$ we have $-K_X \cdot C \geq n+1$. Then $X$ is isomorphic to the projective space.*

*Proof.* Take an unbreakable uniruling $\mathcal{V}$. By 4.5, and our assumption, for a general point $x \in X$ we have that $\mathcal{V}_x$ is smooth and $dim(\mathcal{V}_x) = -K_X \cdot C - 2 = (n-1)$.
By 5.4 and 5.5 and Zariski Main Theorem (birational morphism into a normal scheme has connected fibers), we have that $\mathcal{V}_x \simeq \sigma_\infty \simeq \mathbb{P}^{n-1}$. Let $\tilde{i}_x : \mathcal{V}_x \to \tilde{X} = Bl_x X$ be the lift up of $i_x$; since $Ti_x$ has rank one along $\sigma_\infty$, then $T\tilde{i}_x$ has maximal rank along $\sigma_\infty$, in particular $N_{\sigma_\infty, U_x} \simeq N_{E/\tilde{X}} = O_{\mathbb{P}^{n-1}}(-1)$.
We conclude via an argument of Mori,

Mori's argument as in 3.2 of [Keb02-2].
Consider the Stein factorization of the universal map $i_x : U_x \to X : U_x \to Y \to X$, where the first map $\alpha : U_x \to Y$ contracts the divisor $\sigma_\infty$ and the second $\beta : Y \to X$ is a finite map.
Since $R^1\pi_*(\mathcal{O}_{U_x}) = 0$ and $\mathcal{O}_{U_x(\sigma_\infty)_{|\sigma_\infty}} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, the push forward of the twisted ideal sheaf sequence

$$0 \to \mathcal{O}_{U_x} \to \mathcal{O}_{U_x}(\sigma_\infty) \to \mathcal{O}_{U_x(\sigma_\infty)_{|\sigma_\infty}} \to 0$$

gives on $\mathcal{V}_x \simeq \mathbb{P}^{n-1}$ a sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n-1}} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to 0,$$

where $\mathcal{E}$ is a vector bundle of rank 2 and $U_x \simeq \mathbb{P}(\mathcal{E}^*)$.
Since $Ext^1_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1), \mathcal{O}_{\mathbb{P}^{n-1}}) = 0$, then $U_x \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$. An application of Zariski's main theorem shows that $\alpha$ is the standard contraction of $\sigma_\infty$, that is $Y = \mathbb{P}^n$
We have that adiunction formula for a finite, surjective morphism:

$$-K_{\mathbb{P}^n} = \beta^*(-K_X) + \text{branch divisor.}$$

Let $l$ be a line through $\alpha(x)$ and $t = \beta(l)$; t is a curve associated with $\mathcal{V}_x$. Thus we have

$$n{+}1 = -K_X{\cdot}t = (\beta^*(-K_X)){\cdot}l = (-K_{\mathbb{P}^n}) - \text{branch divisor}){\cdot}l = n{+}1 - \text{branch divisor}{\cdot}l$$

Thus the branch divisor is empty and $\beta$ is birational, thus an isomorphism.

The following generalization of 6.1 is due to Andreatta and Wisniewski [AW01].

**Theorem 6.3.** *Let $X$ be a complex projective manifold of dimension $n \geq 3$. Assume that there exist a subsheaf $E \subset TX$ which is an ample vector bundle. Then $X$ is isomorphic to the projective space.*

*Proof.* By the assumption that there exist a subsheaf $E \subset TX$ which is an ample vector bundle we can apply the Theorem 2.3 and therefore $X$ is uniruled. Take an unbreakable uniruling $\mathcal{V}$.
For a general $f \in \mathcal{V}$ we have $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus d} \oplus \mathcal{O}^{\oplus(n-d-1)}$ where $d = deg(f^*(-K_X)) - 2$ (see 4.13).

**Lemma 6.4.** *For any $f \in \mathcal{V}$ the pull-back $f^*E$ is isomorphic either to $\mathcal{O}(1)^{\oplus r}$ or to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$. In particular the family of curves parametrized by $V$ is unsplit.*

*Proof.* For a general $f \in \mathcal{V}$ the pull-back $f^*E$ is an ample subbundle of $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(d)} \oplus \mathcal{O}^{\oplus(n-d-1)}$ and thus it is as in the lemma. Since $E$ is ample this is true also for all $f \in V$. Since $deg(f^*E) = r$ or $deg(f^*E) = r+1$ and $r > 1$, and for any ample bundle $\mathcal{E}$ over a rational curve we have $deg(\mathcal{E}) \geq rank(\mathcal{E})$, it follows that no curve from $V$ can be split into a sum of two or more rational curves, hence $V$ is unsplit. $\square$

We shall analyze $X$ using the notions of rc$\mathcal{V}$ relation and rc$\mathcal{V}$ fibration. The following is a key observation.

**Lemma 6.5.** *Let $X$, $E$ and $\mathcal{V}$ be as above and moreover assume that $\varphi^0 : X^0 \to Z^0$ is an rc$\mathcal{V}$ fibration. Then $E$ is tangent to a general fiber of $\varphi^0$. That is, if $X_g$ is a general fiber of $\varphi^0$, then the injection $E_{|X_g} \to TX_{|X_g}$ factors via $E_{|X_g} \hookrightarrow TX_g$.*

*Proof.* Choose a general $X_g$ (in particular smooth) and let moreover $x \in X_g$ and $f \in \mathcal{V}_x$ be general as well. Then $Locus(\mathcal{V}_x) \subset X_g$. By 3.0.3 the tangent space to $Locus(\mathcal{V}_x)$ at $f(p)$ is the image of the evaluation of sections of the twisted pull-back of $TX$, which is $= (f^*TX)^+_p$, therefore $(f^*TX)^+_p \subset (f^*TX_g)_p$ for every $p \in \mathbb{P}^1 \setminus \{0\}$. This implies that $E_{|X_g} \to TX_{|X_g}$ factors to $E_{|X_g} \to TX_g$ generically and since the map $TX_g \to TX_{|X_g}$ has cokernel which is torsion free (it is the normal sheaf which is locally free) this yields $E_{|X_g} \hookrightarrow TX_g$, a sheaf injection.

**Proposition 6.6.** *The general fiber of $\varphi^0$, $X_g$, is $\mathbb{P}^k$ and $E_{|X_g} = \mathcal{O}(1)^{\oplus r}$ or $E_{|X_g} = TX_g$.*

*Proof.* By abuse during the proof we denote the general fiber with $X := X_g$. We consider here only the case when for $f \in \mathcal{V}$ the pull-back $f^*E$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$. In particular $f^*E \subset (f^*TX)^+$

Comparing the splitting type of $f^*E$ and $f^*TX$ we see that the tangent map $Tf : T\mathbb{P}^1 \to f^*TX$ factors to a vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \to f^*E$. (In other words, we have surjective morphism $(f^*E)^* \to \Omega_{\mathbb{P}^1} \simeq \mathcal{O}(-2)$).

The vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \to f^*E$ implies $(f^*TX)^+ \hookrightarrow f^*E$. In fact, choose a general $f$ which is an immersion at $0 \to x$. Then $\Phi_x([f]) \in P(E_x) = P((f^*E)_0) \subset P(T_xX) = P((f^*TX)_0)$ and the same holds for morphisms in a neighborhood of $[f]$ in $V_x$. Thus around $\Phi_x([f])$ the tangent cone $S_x$ is contained in $P(E_x) = P((f^*E)_0)$, so is its tangent space $P((f^*TX)_0^+)$ (see 5.7).

Therefore $f^*E = (f^*TX)^+$ and thus $\deg(f^*E) = \deg(f^*(-K_X))$. Since $\rho(X) = 1$ it follows that $\det(E) = -K_X$.

The embedding $E \hookrightarrow TX$ gives rise to a non-trivial morphism $\det(E) \to \Lambda^r TX$ and thus to a non-zero section of $\Lambda^r TX \otimes K_X$. We use dualities to have the equalities:

$$h^0(X, \Lambda^r TX \otimes K_X) = h^n(X, \Omega^r_X) = h^r(X, \Omega^n_X) = h^r(X, K_X) = h^{n-r}(X, \emptyset_X)$$

and, since $X$ is Fano, the latter number is non-zero only if $r = n$. Thus $\Lambda^r TX \otimes (\det E)^{-1} \simeq \mathcal{O}_X$ so $E \hookrightarrow TX$ is nowhere degenerate, hence an isomorphism. We conclude by Theorem 6.1.

Finally we prove that $dim Z_0$ is zero , i.e. $X$ is rationally connected. By contradiction if $dim Z_0 \geq 1$ in [AW01] we proved that :

**Lemma 6.7.** *Outside a subset of codimension $\geq 2$ the morphism $\varphi_0$ is a $\mathbb{P}^k$-bundle (in the analytic topology).*

Then we take a complete curve $B \subset Z_0$ and we consider the $\mathbb{P}^k$-bundle $\varphi_0 : X_B := \varphi_0^{-1}(B) \to B$ with the ample vector bundle $E_{|X_B}$.

We get a contradiction applying the following result, which is due to Campana and Peternell.

**Lemma 6.8.** *Let $X$ be a $n$-dimensional projective manifold, $\varphi : X \to Y$ a $\mathbb{P}^k$ bundle ($k < n$) of the form $X = \mathbb{P}(V)$ with a vector bundle $V$ on $Y$. Then the relative tangent sheaf $T_{X/Y}$ does not contain an ample locally free subsheaf*

## 7. Rational Curves on Symplectic varieties

A holomorphic 2-form $\omega$ on a smooth variety is called **symplectic** if it is closed and non-degenerate at every point.
A **symplectic variety** is a normal variety $Y$ whose smooth part admits a holomorphic symplectic form $\omega_Y$ such that its pull back to any resolution $\pi : X \to Y$ extends to a holomorphic 2-form $\omega_X$ on $X$.
We call $\pi$ a **symplectic resolution** if $\omega_X$ is non degenerate on $X$, i.e. it is a symplectic form.
More generally, a map $\pi : X \to Y$ is called a **symplectic contraction** if $X$ is a symplectic manifold, $Y$ is normal and $\pi$ is a birational projective morphism. If moreover $Y$ is affine we will call $\pi : X \to Y$ a **local symplectic contraction** or **local symplectic resolution**. The following facts are well known.

**Proposition 7.1.** *Let $Y$ be a symplectic variety and $\pi : X \to Y$ be a resolution. Then the following statement are equivalent: (i) $\pi^*K_Y = K_X$, (ii) $\pi$ is symplectic, (iii) $K_X$ is trivial, (iv) for every symplectic form on $Y_{reg}$ its pull-back extends to a symplectic form on $X$.*
*Note that $Y$ is Gorenstein and $K_Y$ is trivial.*

**Corollary 7.2.** *By the Grauert Riemeschneider Teorem $0 = R^i\pi_*K_X = R^i\pi_*\mathcal{O}_X$ for all positve $i$. In particular*
*- $Y$ has rational singularities.*
*- All exceptional fibers of $\pi$ are uniruled.*

**Theorem 7.3.** *Let $\pi : X \to Y$ be a symplectic resolution with $dimX = 2n$. Let also $f : \mathbb{P}^1 \to X$ be a non constant morphism such that $f(\mathbb{P}^1)$ is an $f$-exceptional curve.*
*Then $dim\ Hom_f(\mathbb{P}^1, X) \geq 2n + 1$.*

*Proof.* The Theorem was proved by Z. Ran [Ra95] in the case $X$ is projective. In general it was proved by J. Wierzba [Wie03].
Wierza uses a Theorem of Bogomolov, Beauville, Todorov which says the following:

**Theorem 7.4.** *Let $X$ be a compact symplectic manifold. Then the deformation space (the Kuranishi space) of the complex structure of $X$ is smooth and its tangent space at $[X]$ is excactly $H^1(X, \Omega^1_X)$. Moreover, given a homology class $\alpha \in H_2(X, \mathbb{Q}) = H^{2,0}(X)^*$ which is represented by a rational 1-cycle, there is a one-parameter deformation $\mathcal{Y} = \{Y_t\}_{t\in T}$ such that the flat lifting $\alpha \in H_2(X_t, \mathbb{Q})$ of $\alpha$ is no more an algebraic cycle for general $t \in T$.*

Thus there exists a first order symplectic deformation of $X$, which stays in an unobstructed deformation $\chi$, such that all deformations of $f$ stay in $X$. After showing that all the pertinent deformations are "represented" by algebraic spaces, he shows that $g : \mathbb{P}^1 \to X \subset \chi$ deform in a family of dimension (Mori)

$$dim_g Hom(\mathbb{P}^1, \chi) = \chi(\mathbb{P}^1, g^*T_\chi) \geq dim\chi - degf^*K_\chi \geq 2n + 1.$$

Since all the deformation of $f$ stays in $X$ then $dim\ Hom_f(\mathbb{P}^1, X) = dim\ Hom_g(\mathbb{P}^1, \chi)$ and we are done. $\square$

It follows from this last result the following Theorem (semismall property).

**Theorem 7.5.** *A symplectic resolution $\pi : X \to Y$ is semismall, that is for every closed subvariety $Z \subset X$ we have $2 \operatorname{codim} Z \geq \operatorname{codim} \pi(Z)$. If equality holds $Z$ then is called a maximal cycle.*

*Proof.* Sketch: let $F \subset X$ be a generic fiber of $Z \to \pi(Z)$, let also $d = dimZ$ and $e = dim(\pi(Z))$. We know that all exceptional fibers are uniruled (7.2); take then $\mathcal{V}$ be a generically unsplit family which covers $F$ as in 4.9 and let $V := u^{-1}(\mathcal{V})$. Then by 4.10 we have

$$dimV = dimLocus(V) + dimLocus(V, 0 \to x) + 1 \leq 2dimF + 1 = 2d - 2e + 1.$$

Let $f : \mathbb{P}^1 \longrightarrow F$ be an rational curve in $V$; since $f(\mathbb{P}^1)$ gets contracted under $\pi$, all its deformation in $X$ stay in the exceptional set and we may assume that all small deformations stay in $Z$. Therefore

$$dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, X) = dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, Z) = dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, F) + e \leq 2d - e + 1.$$

By the above Theorem 7.3 we have on the other hand that $dim_{[f]}\mathrm{Hom}(\mathbb{P}^1, X) \geq 2n + 1$ and the Theorem follows. $\square$

## 8. Local symplectic contractions in dimension 4.

8.1. **MDS structure.** In this section $\pi : X \to Y$ is a local symplectic contraction and $\dim X = 4$.
By the semismall property (see Theorem 7.5), the fibers of $\pi$ have dimension less or equal to 2. We will denote with 0 the unique (up to shrinking $Y$ to a smaller affine set) point such that $\dim \pi^{-1}(0) = 2$. If $\pi$ is divisorial then the general non trivial fiber has dimension 1.
By $N_1(X/Y)$ we denote the $\mathbb{Q}$ vector space of 1-cycles proper over $Y$, modulo numerical equivalence (c.f. [KM98, Example 2.16]). Then $N_1(X/Y)$ and $N^1(X/Y)$ are dual via the intersection pairing. Since $R^i\pi_*\mathcal{O}_X = 0$ for $i > 0$, it follows that $N^1(X/Y)$ is a finite dimensional vector space.
The following theorem was proved by Wierzba-Wisniewski, a version in higher dimension has been proved independently by Cho-Miyaoka-Shepherd-Barron. It is a sort of *relative* characterization of the projective space: the hard part is to prove that the two dimensional fiber is normal, then the proof is as in Section 6.

**Theorem 8.1.** *Suppose that $\pi$ is small (i.e. it does not contract any divisor). Then $\pi$ is locally analytically isomorphic to the collapsing of the zero section in the cotangent bundle of $\mathbb{P}^2$. Therefore $X$ admits a Mukai flop*

The above theorem, together with Matsuki's termination of 4-dimensional flops is the key ingredient in the proof of the following result.

**Theorem 8.2.** *Let $\pi : X \to Y$ be a 4-dimensional local symplectic contraction and let $\pi^{-1}(0)$ be its only 2-dimensional fiber. Then $X$ is a Mori Dream Space over $Y$. Moreover any SQM model of $X$ over $Y$ is smooth and any two of them are connected by a finite sequence of Mukai flops whose centers are over $0 \in Y$. In particular, there are only finitely many non isomorphic (local) symplectic resolutions of $Y$,.*

## 9. Rational curves and differential forms

9.1. **The set-up.** In this section $\pi : X \to Y$ will be a local symplectic divisorial contraction and $\dim X = 4$. Call $D \subset X$ the exceptional locus (of dimension 3) and $S = \pi(D) \subset Y$ (of dimension 2).
A general fiber of $\pi$ over any component of $S$ is a configuration of $\mathbb{P}^1$'s with dual graph being a Dynkin diagram (see Theorem 1.3 of [Wie03] for further details).

Choose an irreducible component of $S$, call it $S'$. Take an irreducible curve $C \simeq \mathbb{P}^1$ in a (general) fiber over a point in $S' \setminus \{0\}$ and let $D'$ be the irreducible component of $D$ which contains $C$; note that $\pi(D') = S'$ and $S'$ may be (and usually is) non-normal. Let $\mathcal{V}' \subset Chow(X/Y)$ be an irreducible component of the Chow scheme of $X$ containing $C$. By $\mathcal{V}$ we denote its normalization and $p : \mathcal{U} \to \mathcal{V}$ is the normalized pullback of the universal family over $\mathcal{V}'$. Finally, let $q : \mathcal{U} \to D' \subset X$ be the evaluation map. The contraction $\pi$ determines a morphism $\tilde{\pi} : \mathcal{V} \to S'$, which is surjective because $C$ was chosen in a general fiber over $S'$. We let $\mu : \mathcal{V} \to \widetilde{S}' \to S'$ be its Stein factorization.
We will assume that $\mu$ is not an isomorphism which is equivalent to say that $D'$ has a 2-dimensional fiber over 0.
We will assume that $S'$ is analytically irreducible at 0 or that $\nu^{-1}(0)$ consists of single point. The exceptional locus of $\mu$ is $\mu^{-1}(\nu^{-1}(0)) = \bigcup_i V_i$ where $V_i \subset \mathcal{V}$ are irreducible curves.

$$
\begin{array}{ccccc}
\mathcal{U} & & \xrightarrow{\quad q \quad} & & D' \subset X \\
\downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle \pi} \\
\mathcal{V} & \xrightarrow{\mu} & \widetilde{S}' & \xrightarrow{\nu} & S' \subset Y
\end{array}
\tag{9.1.4}
$$

If necessary, we can take $\mathcal{V}$ to be smooth, eventually by replacing it with its desingularization and $\mathcal{U}$ with the normalized fiber product.

**Theorem 9.1.** *The surface $\widetilde{S}'$ has at most Du Val (or $\mathbb{A} - \mathbb{D} - \mathbb{E}$) singularity at $\nu^{-1}(0)$ and $\mu : \mathcal{V} \to \widetilde{S}'$ is its, possibly non-minimal, resolution. In particular every $V_i$ is a rational curve.*

We note that although the surface $\widetilde{S}'$ is the same for all the symplectic resolutions of $Y$, the parametric scheme for lines, which is a resolution of $\widetilde{S}'$ may be different for different SQM models, see 9.2 for an explicit example.

9.2. **Proof: the differentials.** Let us consider the derivative map $Dq : q^*\Omega_X \to \Omega_{\mathcal{U}}$. We have another derivation map into $\Omega_{\mathcal{U}}$, namely $Dp : p^*\Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}}$. It fits in the exact sequence

$$p^*\Omega_{\mathcal{V}} \longrightarrow \Omega_{\mathcal{U}} \longrightarrow \Omega_{\mathcal{U}/\mathcal{V}} \longrightarrow 0, \tag{9.2.5}$$

whose dual sequence is

$$0 \longrightarrow T_{\mathcal{U}/\mathcal{V}} \longrightarrow T_{\mathcal{U}} \longrightarrow p^*T_{\mathcal{V}} \tag{9.2.6}$$

The symplectic form on $X$, that is $\omega_X$, gives an isomorphism $\omega_X : T_X \to \Omega_X$. We consider the following diagram involving morphism of sheaves over $\mathcal{U}$ appearing in the above sequences.

$$\begin{tikzcd}
T_{\mathcal{U}/\mathcal{V}} \arrow[r] & T_{\mathcal{U}} \arrow[r, "(Dp)^*"] \arrow[d, "(Dq)^*"] & p^*(T_{\mathcal{V}}) \arrow[r, dotted, "p^*(\omega_{\mathcal{V}})"] & p^*(\Omega_{\mathcal{V}}) \arrow[d, "Dp"] & \\
 & q^*T_X \arrow[r, "q^*(\omega_X)"'] & q^*\Omega_X \arrow[r, "Dq"] & \Omega_{\mathcal{U}} \arrow[r] & \Omega_{\mathcal{U}/\mathcal{V}}
\end{tikzcd} \tag{9.2.7}$$

We claim that the dotted arrow exists and it is obtained by a pull back of a two form $\omega_{\mathcal{V}}$ on $\mathcal{V}$, and it is an isomorphism outside the exceptional set of $\mu$ which is $\bigcup_i V_i$.

Indeed, the composition of arrows in the diagram which yields $T_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ is given by the 2-form $Dq(\omega_X)$; it is zero on $T_{\mathcal{U}/\mathcal{V}} \subset T_{\mathcal{U}}$, because this is a torsion free sheaf and its restriction to any fiber of $p$ outside $\bigcup_i V_i$ (any fiber of $p$ is there a $\mathbb{P}^1$) is $\mathcal{O}(2)$ while the restriction of $\Omega_{\mathcal{U}}$ is $\mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}$. Therefore we have that it is in fact a map $p^*T_{\mathcal{V}} \to \Omega_{\mathcal{U}}$.

By the same reason the composition $T_{\mathcal{U}} \to \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}/\mathcal{V}}$ is zero since $T_{\mathcal{U}}$ on any fiber of $p$ outside $\bigcup_i V_i$ is $\mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O}$ while $\Omega_{\mathcal{U}/\mathcal{V}}$ is $\mathcal{O}(-2)$. Thus the map $Dq(\omega_X) : p^*T_{\mathcal{V}} \to \Omega_{\mathcal{U}}$ factors through $p^*(T_{\mathcal{V}}) \to p^*(\Omega_{\mathcal{V}})$.

As a result, since it is trivial on the fiber of $p$, $Dq(\omega_X) = Dp(\omega_{\mathcal{V}})$, for some 2-form $\omega_{\mathcal{V}}$ on $\mathcal{V}$.

Since $Dq$ is of maximal rank outside of $p^{-1}(\bigcup_i V_i)$ and $p$ is just a $\mathbb{P}^1$-bundle there, it follows that $\omega_{\mathcal{V}}$ does not assume zero outside the exceptional set of $\mu$. Hence $K_{\mathcal{V}} = \sum a_i V_i$, with $a_i \geq 0$ being the discrepancy of $V_i$.

9.3. **Examples: quotient symplectic singularities.** Let $S$ be a smooth surface (proper or not). Denote by $S^{(n)}$ the *symmetric product* of $S$, that is $S^{(n)} = S^n/\sigma_n$, where $\sigma_n$ is the symmetric group of peremutations of $n$ elements. Let also $Hilb^n(S)$ be the *Hilbert scheme* of 0-cycles of degree $n$. A classical result says that $Hilb^n(S)$ is smooth and that $\tau : Hilb^n(S) \to S^{(n)}$ is a crepant resolution of singularities. We will call it a Hilb-Chow map.
Suppose now that $S \to S'$ is a resolution of a Du Val singularity which is of type $S' = \mathbb{C}^2/H$ with $H < SL(2, \mathbb{C})$ a finite group. Then the composition $Hilb^n(S) \to S^{(n)} \to (S')^{(n)}$ is a local symplectic contraction.
We note that $(S')^{(n)}$ is a quotient singularity with respect to the action of the wreath product $H \wr \sigma_n = (H^n) \rtimes \sigma_n$ (the group $\sigma_n$ permutes factors in $H^n = H^{\times n}$).
Let consider the case $n = 2$, i.e. let $H < SL(2)$ be a finite subgroup and let $G := H^{\times 2} \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ interchanges the factors in the product. We write $G = H \wr \mathbb{Z}_2$.
Note that $\mathbb{Z}_{n+1} \wr \mathbb{Z}_2$ has another nice presentation, namely $(\mathbb{Z}_{n+1})^{\times 2} \rtimes \mathbb{Z}_2 = D_{2n} \rtimes \mathbb{Z}_n$, where $D_{2n}$ is the dihedral group of the regular $n$-gon and $\mathbb{Z}_n$ acts on it by rotations.
We consider the projective symplectic resolution described above:

$$\pi : X := Hilb^2(S) \to S^{(2)} \to (\mathbb{C}^2/H)^{(2)} := Y$$

where $\nu : S \to \mathbb{C}^2/H$ is the minimal resolution with the exceptional set $\bigcup_i C_i$, where $C_i$, $i = 1, ..., k$, are $(-2)$-curves.
The morphism $\tau : Hilb^2(S) \to S^{(2)}$ is just a blow-up of the locus of $\mathbb{A}_1$ singularities (the image of the diagonal under $S^2 \to S^{(2)}$) with irreducible exceptional divisor $E_0$ which is a $\mathbb{P}^1$ bundle over $S$. We set $S' = \pi(E_0)$. By $E_i$, with $i = 1, \ldots, k$ we denote the strict transform, via $\tau$, of the image of $C_i \times S$ under the map $S^2 \to S^{(2)}$. By $e_i$ we denote the class of an irreducible component of a general fiber of $\pi_{|E_i}$. The image $\pi(E_i)$ for $i \geq 1$ is the surface $S'' \simeq C^2/H$. The singular locus of $Y$ is the union $S = S' \cup S''$.
The irreducible components of $\pi^{-1}(0)$ are described in the following.

- $P_{i,i}$, for $i = 1, ..., k$. They are the strict transform of $C_i^{(2)}$ via $\tau$. They are isomorphic to $\mathbb{P}^2$.
- $P_{i,j}$, for $i, j = 1, ..., k$ and $i < j$. They are the strict transform via $\tau$ of the image of $C_i \times C_j$ under the morphism $S^2 \to S^{(2)}$. They are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if $C_i \cap C_j = \emptyset$ and to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ if $C_i \cap C_j \neq \emptyset$.
- $Q_i$, for $i = 1, ..., k$. They are the preimage $\tau^{-1}(\Delta_{C_i})$, where $\Delta_{C_i}$ is the diagonal embedding of $C_i$ in $S^{(2)}$. They are isomorphic to $\mathbb{P}(T_{S|C_i}) = \mathbb{P}(\mathcal{O}_{C_i}(2) \oplus \mathcal{O}_{C_i}(-2))$, i.e. to the Hirzebruch surface $F_4$.

The Figure 1 presents a "realistic" description of configurations of components in the special fiber of symplectic resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$. By abuse, the strict transforms of the components and the results of the flopping of $\mathbb{P}^2$'s are denoted by the same letters.
The position of these configurations in Figure 1 is consistent with the decomposition of the cone $\mathrm{Mov}(X/Y)$. In particular, the configuration at the top is associated with the Hilbert-Chow resolution. Note that the central configuration of this diagram cointains three copies of $\mathbb{P}^2$, denoted $P_{ij}$, which contain lines whose classes are $e_0 - e_1$, $e_0 - e_2$ and $e_1 + e_2 - e_0$.
On the other hand, the configuration in the bottom is associated with the resolution which can be factored by two different divisorial elementary contractions of classes

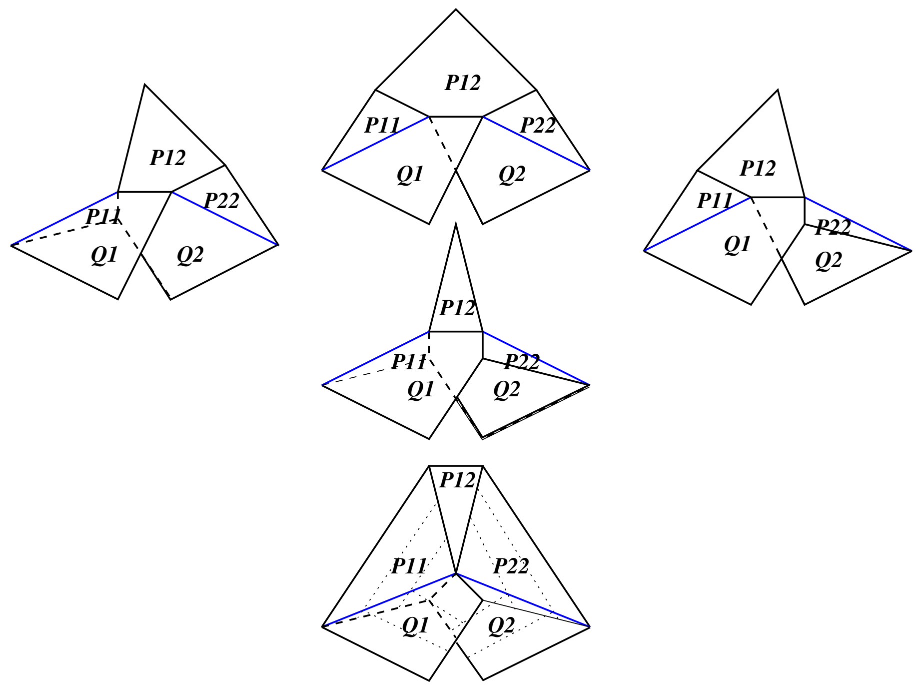


FIGURE 1. Components of the central fiber in resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$

$e_1$ and $e_2$. In fact, contracting both $e_1$ and $e_2$ is a resolution of $\mathbb{A}_2$ singularities which is a part of a resolution of $Y$ which comes from presenting $\mathbb{Z}_3 \wr \mathbb{Z}_2$ as $D_6 \rtimes \mathbb{Z}_3$. That is, $X$ is then obtained by first resolving the singularities of the action of $D_6 = \sigma_3$ and then by resolving the singularities of the $\mathbb{Z}_3$ action on this resolution. The rulings of the respective surfaces coming from this last blow up are indicated by dotted line segments. We will call such $X$ a $D_6 \rtimes \mathbb{Z}_3$-resolution.

By $\mathcal{V}_0$ we denote the component of $Chow(X/Y)$ dominating $S'$ and parametrizing curves equivalent to $e_0$, while by $\mathcal{V}_1$ and $\mathcal{V}_2$ we denote components dominating $S''$ parameterizing deformations of $e_1$ and $e_2$. The surfaces $\mathcal{V}_i$ may depend on the resolution and, in fact, while $\mathcal{V}_1$ and $\mathcal{V}_2$ remain unchanged, the component $\mathcal{V}_0$ will change under flops.

**Lemma 9.2.** *If $X$ is the Hilbert-Chow resolution then $\mathcal{V}_0$ is the minimal resolution of $\mathbb{A}_2$ singularity. If $X$ is the $D_6 \rtimes \mathbb{Z}_3$-resolution then $\mathcal{V}_0$ is non-minimal, with one $(-1)$ curve in the central position of three exceptional curves.*

*Proof.* The first statement is immediate. To see the second one, note that we have the map of $\mathcal{V}_0$ to Chow of lines in the resolution of $\mathbb{C}^4/\sigma_3$ divided by $\mathbb{Z}_3$ action. The $\mathbb{Z}_3$-action in question is just a lift up of the original linear action on the fixed point set of rotations in $\sigma_3 = D_6$ hence $\mathcal{V}_0$ resolves 2 cubic cone singularities associated with the eigenvectors of the original action. $\square$

One may verify that in the $D_6 \rtimes \mathbb{Z}_3$-resolution the exceptional set in $\mathcal{V}_0$ parametrizes curves consisting of three components: $\mathbb{Q}_2 \cap P_{11}$, $Q_1 \cap P_{22}$ and a line in $P_{12}$, whose classes are, respectively, $e_2$, $e_1$ and $e_0 - (e_1 + e_2)$.

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