2 Existence and uniqueness theorems for the Cauchy problem

2.1 Definitions: local and global existence

Let us consider the following Cauchy problem for a nonlinear nonautonomous system[27]

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = x_0, \end{cases} \tag{2.1}$$

where $(t_0, x_0) \in A \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ is fixed, with A a given open set and $f : A \to \mathbb{R}^n$ continuous.

Definition 2.1 *We say that a function $y : I \to \mathbb{R}^n$ is a <u>solution of (2.1) on I</u> if $I \subseteq \mathbb{R}$ is an open interval, $t_0 \in I$, $y \in C^1(I, \mathbb{R}^n)$, $y'(t) = f(t, y(t))$ for all $t \in I$ and $y(t_0) = x_0$.*

Note that Definition 2.1 also implies that $(t, y(t)) \in A$ for all $t \in I$, otherwise we cannot apply f to the couple $(t, y(t))$. Moreover, it is evident that, if $y : I \to \mathbb{R}^n$ is a solution on I, then it is also a solution on every open interval $J \subseteq I$ containing t_0.

Definition 2.2 *We say that the problem (2.1) <u>admits a local solution</u> (or equivalently <u>is locally solvable</u>) if there exist an open interval $I \subseteq \mathbb{R}$ and a function $y : I \to \mathbb{R}^n$ such that y is a solution of (2.1) on I.*

Definition 2.3 *We say that the problem (2.1) <u>admits global solutions</u> (or equivalently <u>is globally solvable</u>) if for every open interval $I \subseteq \mathbb{R}$ such that*

$$t_0 \in I, \qquad \{x \in \mathbb{R}^n | (t, x) \in A\} \neq \emptyset \quad \forall\, t \in I, \tag{2.2}$$

there exists a function $y : I \to \mathbb{R}^n$ which is solution of (2.1) on I.

The difference between local and global solvability is that in the first case we cannot a-priori fix the time interval, but we only know that a (possibly very small) time interval exists. On the contrary, in the second case, for any (suitable) fixed time interval, we find a solution on it.

Of course, the global solvability implies the local one, but the contrary is false.

Example 2.4 We consider the problem

$$\begin{cases} y'(t) = y^2(t), \\ y(0) = 1. \end{cases}$$

Here, $n = 1$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(t, x) = x^2$, $A = \mathbb{R}^2$, $t_0 = 0$, $x_0 = 1$. Applying a uniqueness result from next subsections, we see that such a problem has the unique solution

[27] Here "nonlinear system" means that we do not require that the system is linear. But, of course, all the results can also be applied to the linear case. The same remark applies to "nonautonomous system".

$$y(t) = \frac{1}{1-t},$$

which is defined only in the time interval $]-\infty, 1[$. For example, if we a-priori fix the time interval $J =]-1, 2[$ (which is an admissible time interval for our problem, that is satisfies (2.2)), and we look for a solution on J, then we do not find anything. Hence the problem admits only a local solution.

In general, if A is arch-connected, we may consider the maximal interval satisfying (2.2)

$$I_{\max} = \{t \in \mathbb{R} | \exists x \in \mathbb{R}^n, (t, x) \in A\} =:]a, b[$$

Moreover, let us suppose that A is the maximal set of definition for f and that the problem is only locally solvable. Then there exist $a < \xi_1 < \xi_2 < b$ such that every solution exits from A[28] in the time interval $]\xi_1, \xi_2[$. Hence, looking for solution defined up to times $t \in]a, \xi_1[\cup]\xi_2, b[$ is meaningless since we certainly cannot apply f to the possible couple $(t, y(t)) \notin A$.

2.2 The integral representation

Proposition 2.5 *Let $f : A \to \mathbb{R}^n$ be continuous and $I \subseteq \mathbb{R}$ be an open interval. Then a function $y : I \to \mathbb{R}^n$ is a solution of (2.1) on I, if and only if $(t, y(t)) \in A$ for all $t \in I$, and, more important, y is continuous (i.e. $y \in C(I, \mathbb{R}^n)$) and*[29]

$$y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))ds \qquad \forall t \in I \tag{2.3}$$

Proof. First of all let us note that, since y and f are both continuous, then the function of time $s \mapsto f(s, y(s))$ is also continuous and the integral in (2.3) exists for all $t \in I$. Let us now prove the equivalence.

$\Longrightarrow$) If y is a solution then, by definition, $y \in C^1(I, \mathbb{R}^n)$ and so, for every $t \in I$, integrating the equation from t_0 to t, we get

$$\int_{t_0}^{t} y'(s)ds = \int_{t_0}^{t} f(s, y(s))ds \implies y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))ds.$$

$\Longleftarrow$) If y is continuous and satisfies (2.3), then we also have $y \in C^1(I, \mathbb{R}^n)$, and, by the fundamental theorem of calculus, its derivative is exactly $y'(t) = f(t, y(t))$. Since by (2.3) we obviously have $y(t_0) = x_0$, we then conclude. $\square$

[28] By an abuse of definition, this means that the couple $(t, y(t)) \in \mathbb{R}^{n+1}$ exits from A.

[29] The integral of a vectorial function f is the vector whose components are the integral of the components of f.

Proposition 2.5 gives an integral representation of the solution, as well as of the ordinary differential equation. It is a powerful tool for studying the Cauchy problem, since integrals are often more manageable than derivatives. Already starting from the next subsection we are going to make an intensive use of the integral representation (2.3).

Remark 2.6 *Note that the formula (2.3) is an "implicit formula". That is we cannot calculate the solution y just integrating in (2.3), because the second member itself depends on y.*

2.3 The Peano existence theorem under continuity hypothesis

Theorem 2.7 *If $f : A \to \mathbb{R}^n$ is continuous, then the Cauchy problem (2.1) is locally solvable for any choice of the datum $(t_0, x_0) \in A$. This means that, for any $(t_0, x_0) \in A$, there exist $\delta > 0$ (depending on (t_0, x_0)) and a function $y :]t_0 - \delta, t_0 + \delta[\to \mathbb{R}^n$ which is a solution of (2.1) on $]t_0 - \delta, t_0 + \delta[$.*

In the next two subsections we are going to give proofs of this theorem for the particular case where

$$A = J \times \mathbb{R}^n, \ J =]a, b[\subseteq \mathbb{R} \text{ open interval }, \quad f \text{ bounded on } A, \tag{2.4}$$

and we prove the existence of a solution in a right neighborhood of t_0: $[t_0, c[$. This is indeed a (right-) global existence result (for the arbitrariness of $t_0 < c < b$). The general case of local (both sides-) existence with arbitrary A, and f not bounded, may be obtained restricting the problem to a rectangular inside A[30].

In both following proofs, the idea is to approximate a solution by constructing a suitable sequence of quasi-solutions, and then passing to the limit using the Ascoli-Arzelà theorem 8.10.

2.3.1 A first proof: a delayed argument

If we look to the equation $y'(t) = f(t, y(t))$, we have an instantaneous relation (i.e. at the same instant t) between the left- and right-hand sides. Hence, we decide to little change the equation by introducing a time-delay $\alpha > 0$. The idea is to consider the delayed equation

$$y'(t) = f(t, y(t - \alpha)),$$

where the unknown term $y(t - \alpha)$ in the right-hand side may be supposed already known, indeed our goal is to solve the equation step-by-step, that is first in the interval $[t_0, t_0 + \alpha]$, then in $[t_0 + \alpha, t_0 + 2\alpha]$ and so on.

We define the following function in $[t_0 - \alpha, c]$

[30] In a similar way as we are going to do in the next Section under Lipschitz continuity hypothesis.

$$z_\alpha(t) = \begin{cases} x_0 & \text{if } t_0 - \alpha \le t \le t_0, \\ x_0 + \displaystyle\int_{t_0}^{t} f(s, z_\alpha(s-\alpha))ds & \text{if } t_0 \le t \le c. \end{cases}$$

Note that, for every $\alpha > 0$, the function z_α is well defined in $[t_0, c]$. For instance: if $t \in [t_0, t_0 + \alpha]$ then $z_\alpha(t) = x_0 + \int_{t_0}^{t} f(s, x_0)ds$, which is computable, and once we know z_α in $[t_0, t_0 + \alpha]$, we then can compute it in $[t_0 + \alpha, t_0 + 2\alpha]$, and so on. Since f is bounded, we have that, denoting by $M > 0$ its bound,

$$\|z_\alpha\|_{C([t_0,c];\mathbb{R}^n)} \le \|x_0\|_{\mathbb{R}^n} + M(c - t_0), \quad \|z_\alpha(t) - z_\alpha(s)\|_{\mathbb{R}^n} \le M|t - s| \ \forall t, s \in [t_0, c]. \quad (2.5)$$

Hence, the functions $\{z_\alpha\}_{\alpha>0}$ are equibounded and equicontinuous on the compact set $[t_0, c]$. By the Ascoli-Arzelà theorem, this implies that there exist a subsequence z_k (i.e. a subsequence $\alpha_k \to 0$ as $k \to +\infty$) and a continuous function $y : [t_0, c] \to \mathbb{R}^n$ such that z_k uniformly converges to y on $[t_0, c]$.

Note that also the functions $t \mapsto z_\alpha(t - \alpha)$ are well defined on $[t_0, c]$, and they also uniformly converge to y. This is true since we have

$$\|z_k(t - \alpha_k) - y(t)\| \le \|z_k(t - \alpha_k) - z_k(t)\| + \|z_k(t) - y(t)\|,$$

and both terms in the right-hand side are infinitesimal as $k \to +\infty$ independently on $t \in [t_0, c]$ (for the uniform convergence and the equicontinuity of z_k (2.5)).

Moreover, the functions $t \mapsto f(t, z_k(t - \alpha_k))$ uniformly converges to the function $t \mapsto f(t, y(t))$, on $[t_0, c]$. This is true since the couples $(t, z_k(t - \alpha_k)), (t, y(t))$ all belong to the compact set $[t_0, c] \times \overline{B}_{\mathbb{R}^n}(O, (\|x_0\|_{\mathbb{R}^n} + M(c - t_0)))$, where O is the origin of $\mathbb{R}^n$, and f, being continuous, is uniformly continuous on such a set.

Finally, passing to the limit, as $k \to +\infty$, in the definition of z_k, we get

$$y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))ds,$$

which concludes the proof, proving that y is a solution.

2.3.2 A (sketched) second proof: the Euler method

Here, our goal is to construct an approximate solution y_δ by dividing the time interval $[t_0, c]$ into a series of a finite number of small subintervals $[t_k, t_{k+1}]$ and, on every such subintervals, defining y_δ as the segment starting from $y_\delta(t_k)$ and parallel to $f(t_k, y_\delta(t_k))$[31] (which may be considered as known if we apply such a procedure step by step (i.e. subinterval by subinterval)). The idea is then to make the limit as δ, the length of the subintervals, goes to zero.

[31] i.e. the linear function with slope given by $f(t_k, y_\delta(t_k))$

Let us take $\delta > 0$ and a finite set of instants

$$\mathcal{T}_\delta = \{t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = c\},$$

such that $t_{k+1} - t_k = \delta$ for all $k = 0, \ldots, m$ and $c - t_m \leq \delta$.

We define the following function $y_\delta : [t_0, c[\to \mathbb{R}^n$

$$y_\delta(t) = y_\delta(t_{k-1}) + f(t_{k-1}, y_\delta(t_{k-1}))(t - t_{k-1}) \quad t \in [t_{k-1,t_k}] \ \forall k = 1, \ldots, m, m+1$$

This is a good definition and y_δ is a polygonal. Note that, in every subinterval $[t_{k-1}, t_k]$, $y'_\delta(t) = f(t_{k-1}, y_\delta(t_{k-1}))$, and hence, also in this case, we have a sort of "delay".

It can be (sufficiently easily) proved that, for every $\varepsilon > 0$, there exists $0 < \delta_\varepsilon < \varepsilon$ (i.e. a sufficiently fine partition of $[t_0, c]$), such that

$$\left\| y_{\delta_\varepsilon}(t) - x_0 + \int_{t_0}^{t} f(\tau, y_{\delta_\varepsilon}(\tau)) d\tau \right\|_{\mathbb{R}^n} \leq \varepsilon \quad \forall t \in [t_0, c], \tag{2.6}$$

which, recalling the integral representation of the solution, may be seen as a criterium of goodness of the approximate solution. Moreover, in the same way as in the previous paragraph, we can prove that the sequence of functions

$$\varphi_\delta : t \mapsto x_0 + \int_{t_0}^{t} f(\tau, y_\delta(\tau)) d\tau$$

is equibounded and equicontinuous in $[t_0, c]$. Hence, also the sequence $\varphi_{\delta_\varepsilon}$ has the same properties and so, by the Ascoli-Arzelà theorem, there exists a subsequence $\varphi_{\delta_{\varepsilon(i)}}$ which, as $i \to +\infty$ (i.e. $\varepsilon(i) \to 0$), uniformly converges in $[t_0, c]$ to a continuous function φ. But then, $y_{\delta_{\varepsilon(i)}}$ also uniformly converges to φ. Indeed, for every $t \in [t_0, c]$, we have[32]

$$\|y_{\delta_{\varepsilon(i)}}(t) - \varphi(t)\| \leq \|y_{\delta_{\varepsilon(i)}}(t) - \varphi_{\delta_{\varepsilon(i)}}(t)\| + \|\varphi_{\delta_{\varepsilon(i)}}(t) - \varphi(t)\| \leq \varepsilon(i) + O(\varepsilon(i))$$

From this we deduce that, at the limit (passing to the limit inside the integral, noting that y_δ are equibounded by $\tilde{M}$ (since f is bounded), and that f is uniformly continuous on $[t_0, c] \times B_{\mathbb{R}^n}(0, \tilde{M})$), we have

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(\tau, \varphi(t)) d\tau \quad \forall\ t \in [t_0, c],$$

and the proof is concluded: we have found a solution (i.e. φ) of the Cauchy problem.

[32]Here O means an infinitesimal with respect to its argument.

2.3.3 An example of non existence

Here, we give an example where the dynamics is discontinuous and the Cauchy problem has no solution at all. Let us consider $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -1 & \text{if } x > 0, \\ \dfrac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x < 0, \end{cases}$$

and consider the Cauchy problem

$$\begin{cases} y'(x) = f(y(x)), \\ y(0) = 0. \end{cases}$$

It is evident that such a problem cannot have a solution, even locally. Indeed, if a solution y exists (remember that we require $y \in C^1$) then $y'(0) = f(y(0)) = f(0) = 1/2 > 0$, and hence y is strictly increasing at $x = 0$. Since $y(0) = 0$, this means that $y > 0$ immediately to the right of $x = 0$, but this is impossible since when $y > 0$ the derivative y' must be negative.

Of course, if we change the value of f at $x = 0$, defining $f(0) = 0$, then f is still discontinuous but the Cauchy problem has a solution, even unique: the null function $y(x) \equiv 0$.

2.3.4 An example of non uniqueness

Let us consider the following scalar Cauchy problem

$$\begin{cases} y'(t) = (y(t))^{\frac{2}{3}}, \\ y(0) = 0. \end{cases}$$

Here, $f(t, x) = x^{2/3}$ which is defined on $A = \mathbb{R} \times \mathbb{R}$ and is continuous. Hence we have, at least, one local solution. Indeed, we have infinitely many local solutions.

First of all, let us note that the null function $y(t) \equiv 0$ is a solution. But, for every $\alpha, \beta \geq 0$, we also have the solution

$$y_{\alpha,\beta}(t) = \begin{cases} 0 & \text{if } -\beta \leq t \leq \alpha, \\ \left(\dfrac{t-\alpha}{3}\right)^3 & \text{if } t \geq \alpha, \\ \left(\dfrac{t+\beta}{3}\right)^3 & \text{if } t \leq -\beta. \end{cases}$$

2.4 Local existence and uniqueness under Lipschitz continuity hypothesis

Definition 2.8 *We say that the Cauchy problem (2.1) has a unique local solution if there exists an interval $\tilde{I}$ and a solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$ on $\tilde{I}$ such that, for every other solution $y : I \to \mathbb{R}^n$, we have $y = \tilde{y}$ on $I \cap \tilde{I}$*[33].

Note that in the Definition 2.8 we suppose that a local solution exists and we say that it is unique in the sense that, in a suitable neighborhood of t_0, all possible solutions must coincide with $\tilde{y}$.

Definition 2.9 *The continuous function $f : A \to \mathbb{R}^n$ of the problem (2.1) is said to be locally Lipschitz continuous in $x \in \mathbb{R}^n$ uniformly with respect to $t \in \mathbb{R}$ if for every $(t_0, x_0) \in A$ there exist a neighborhood of (t_0, x_0), $U \subseteq A$, and a constant $L > 0$ such that*

$$(t, x), (t, z) \in U \implies \|f(t, x) - f(t, z)\|_{\mathbb{R}^n} \leq L\|x - z\|_{\mathbb{R}^n}. \tag{2.7}$$

Theorem 2.10 *(Local existence and uniqueness) If $f : A \to \mathbb{R}^n$ is continuous and satisfies the local uniform Lipschitz condition (2.7), then, for every datum $(t_0, x_0) \in A$, the Cauchy problem (2.1) has a unique local solution in the sense of Definition 2.8.*

Proof. Let $(t_0, x_0) \in A$ be fixed. Our goal is to find $\delta_1 > 0$ and $\delta_2 > 0$ such that, for every $0 < \delta' \leq \delta_1$, the operator

$$\begin{aligned} &T_{\delta'} : C\left([t_0 - \delta', t_0 + \delta']; \overline{B}(x_0, \delta_2)\right) \to C\left([t_0 - \delta', t_0 + \delta']; \overline{B}(x_0, \delta_2)\right), \\ &v \mapsto T_{\delta'}[v] : t \mapsto x_0 + \int_{t_0}^{t} f(s, v(s))ds \end{aligned} \tag{2.8}$$

is well defined and has a unique fixed point. In this way, via the integral representation of solutions (2.3), for every $0 < \delta' \leq \delta_1$, we simultaneously have existence of a solution on $]t_0 - \delta', t_0 + \delta'[$ and uniqueness on such intervals.[34]. Hence, the local uniqueness in the sense of Definition 2.8 comes from the following observation: let us take $0 < \tilde{\delta} \leq \delta_1$, define $\tilde{I} =]t_0 - \tilde{\delta}, t_0 + \tilde{\delta}[$ and define $\tilde{y}$ as the unique solution on $\tilde{I}$. Note that every solution is also a solution on every subinterval. Then, if $y : I \to \mathbb{R}^n$ is a solution, we must have $\tilde{y} = y$ on $\tilde{I} \cap I$ ($\subseteq]t_0 - \delta', t_0 + \delta'[$ for some $0 < \delta' < \tilde{\delta}$).

Hence, we are going to analyze the following four steps.

1) Find $\delta_1 > 0$ and $\delta_2 > 0$ such that, for every $0 < \delta' \leq \delta_1$, $T_{\delta'}$ is well defined, that is $T_{\delta'}(v) \in C\left([t_0 - \delta', t_0 + \delta']; \overline{B}(x_0, \delta_2)\right)$ for all $v \in C\left([t_0 - \delta', t_0 + \delta']; \overline{B}(x_0, \delta_2)\right)$.

2) Recognize that $y :]t_0 - \delta', t_0 + \delta'[\to \mathbb{R}^n$ is a solution of (2.1) on $]t_0 - \delta', t_0 + \delta'[$ if and only if it is a fixed point of $T_{\delta'}$.

3) Take δ_1 as in 1) and also such that, for every $0 < \delta' \leq \delta_1$, $T_{\delta'}$ is a contraction with respect to the infinity norm.

[33] Note that, since I and $\tilde{I}$ are both open intervals containing t_0, then $I \cap \tilde{I}$ is also an open interval containing t_0.

[34] This means that there exists only one solution on $]t_0 - \delta', t_0 + \delta'[$, for every δ'.

4) Apply the Contraction Lemma to $T_{\delta'}$ to observe that it has one and only one fixed point. Conclude by the point 2).

1) Let U and L be as in (2.7). Let us take $\overline{\delta}_1, \delta_2 > 0$ such that $[t_0 - \overline{\delta}_1, t_0 + \overline{\delta}_1] \times \overline{B}(x_0, \delta_2) \subseteq U$, and take $M > 0$ such that[35]

$$\|f(t,x)\| \leq M, \quad \forall\ (t,x) \in [t_0 - \overline{\delta}_1, t_0 + \overline{\delta}_1] \times \overline{B}(x_0, \delta_2).$$

For every $0 < \delta' \leq \overline{\delta}_1$, for every $v \in C\left([t_0 - \delta', t_0 + \delta'], \overline{B}(x_0, \delta_2)\right)$, and for every $t \in [t_0 - \delta', t_0 + \delta']$, we have the following estimate

$$\|T_{\delta'}[v](t) - x_0\| = \left\| \int_{t_0}^{t} f(s, v(s)) ds \right\| \leq \left| \int_{t_0}^{t} \|f(s, v(s))\| ds \right| \leq \left| \int_{t_0}^{t} M ds \right| \leq M\delta' \leq M\overline{\delta}_1. \tag{2.9}$$

Hence, if we take $\delta_1 \leq \overline{\delta}_1$ such that $M\delta_1 \leq \delta_2$, then the point 1) is done[36].

2) Obvious by Proposition 2.5.

3) Let us take $0 < \delta' \leq \delta_1$. For every $u, v \in C\left([t_0 - \delta', t_0 + \delta'], \overline{B}(x_0, \delta_2)\right)$, and for every $t \in [t_0 - \delta', t_0 + \delta']$ we have

$$\begin{aligned} &\|T_{\delta'}[u](t) - T_{\delta'}[v](t)\| \leq \left| \int_{t_0}^{t} \|f(s, u(s)) - f(s, v(s)) ds \right| \leq L \left| \int_{t_0}^{t} \|u(s) - v(s)\| ds \right| \\ &\leq L \left| \int_{t_0}^{t} \|u - v\|_\infty ds \right| \leq L\delta' \|u - v\|_\infty \leq L\delta_1 \|u - v\|_\infty. \end{aligned}$$

By the arbitrariness of $t \in [t_0 - \delta', t_0 + \delta']$, and also because the last term in the right is independent from t, we then get

$$\|T_{\delta'}[u] - T_{\delta'}[v]\|_\infty \leq L\tilde{\delta}_1 \|u - v\|_\infty,$$

where $\| \cdot - \cdot \|_\infty$ is the metrics in $C\left([t_0 - \delta', t_0 + \delta'], \overline{B}(x_0, \delta_2)\right)$.
Hence, if we take δ_1 as in point 1) and also such that $L\delta_1 < 1$, then point 3) is also done.

4) Immediate. □

Remark 2.11 *A sufficient condition for satisfying the local Lipschitz hypothesis (2.7) is to be a C^1 function. Hence, every Cauchy problem with C^1 dynamics f has a unique local solution for every initial datum.*

[35] Such a M exists for the continuity of f and the compactness of $[t_0 - \overline{\delta}_1, t_0 + \overline{\delta}_1] \times \overline{B}(x_0, \delta_2)$.

[36] Also observe that, just by definition, $T_{\delta'}[v]$ is continuous.

2.5 Global existence and uniqueness under Lipschitz continuity hypothesis

Definition 2.12 *We say that the Cauchy problem (2.1) <u>has a unique global solution</u> if <u>it has exactly one solution</u>, in the sense that there exists a solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$, such that for every other solution $y : I \to \mathbb{R}^n$ we have: $I \subseteq \tilde{I}$ and $y = \tilde{y}$ on I.*

Note that Definition 2.12 also requires that a local solution exists. However, the global uniqueness is different from the local uniqueness of Definition 2.8. Indeed, in the second case we require that a local solution $\tilde{y}$ exists in a suitable interval $\tilde{I}$ and any other solution is equal to $\tilde{y}$ in the common instants of existence, but they may be different outside the common instants. On the other hand, the global uniqueness requires that a solution $\tilde{y}$ exists on an interval $\tilde{I}$, that no other solution may be defined for instants $t \notin \tilde{I}$, and that on the common instants of existence (necessarily contained in $\tilde{I}$) any other solution must be equal to $\tilde{y}$.

Of course, the global uniqueness of the solution implies the local one, but the contrary is false.

Example 2.13 The following Cauchy problem has a unique local solution, but not a unique global solution:

$$\begin{cases} y'(t) = (y(t))^{\frac{2}{3}}, \\ y(-1) = -\dfrac{1}{27}. \end{cases}$$

The following two functions are both solution[37]

$$y_1(t) = \begin{cases} \left(\dfrac{t}{3}\right)^3 & t \leq 0, \\ 0 & t \geq 0 \end{cases}, \qquad y_2(t) = \left(\frac{t}{3}\right)^3 \ \forall\, t \in \mathbb{R}.$$

Hence, such a problem has no global uniqueness since y_1 and y_2 are both solution on $I = \mathbb{R}$ but they are different. On the contrary, it has a unique local solution $y(t) = (t/3)^3$, as can be seen by Theorem 2.10.

Remark 2.14 *The global uniqueness in the sense of Definition 2.12 does not imply that a global solution exists, that is that the problem is globally solvable in the sense of Definition 2.3. Indeed, the interval $\tilde{I}$ is not necessarily the maximal interval satisfying the condition (2.2). For instance, the problem in Example 2.4 has a unique global solution in the sense of Definition 2.12, but it is not globally solvable in the sense of Definition 2.3.*

Theorem 2.15 *(Global existence and uniqueness on a strip) Let us consider the Cauchy problem (2.1), with A given by the strip*

[37] See the example in Subsection 2.3.4.

$$A =]a, b[\times \mathbb{R}^n, \tag{2.10}$$

where $-\infty \leq a < b \leq +\infty$. *If* f *satisfies the global uniform Lipschitz condition*

$$\begin{aligned} &\exists\ L > 0 \text{ such that } \forall\ t \in]a, b[,\ \forall\ x, y \in \mathbb{R}^n : \\ &\|f(t, x) - f(t, y)\|_{\mathbb{R}^n} \leq L\|x - y\|_{\mathbb{R}^n}. \end{aligned} \tag{2.11}$$

then, for every datum $(t_0, x_0) \in A$, *the problem has a unique global solution, in the sense of Definition 2.12. Moreover the solution is defined on the whole interval* $]a, b[$, *that is the problem is globally solvable.*

Proof. We are going to prove that on every subinterval of $]a, b[$ there exists one and only one solution, from which the conclusion will follow.

Let us refer to the proof of Theorem 2.10. Over there, given (t_0, x_0), we have taken the neighborhood U where we have found the local Lipschitz constant L (both depending on (t_0, x_0)); we have taken $\overline{\delta}_1$ and δ_2 depending on (t_0, x_0), and moreover we have taken M also depending on (t_0, x_0). All those choices have leaded us to choose

$$\delta_1 = \min\left\{\overline{\delta}_1, \frac{\delta_2}{M}, \frac{1}{L}\right\},$$

in order to prove the statement. Of course, δ_1 was also depending on (t_0, x_0), and recall that it was the semi-amplitude of the existence interval for the unique local solution. Now, our goal is to construct δ_1 independently from (t_0, x_0).

Let us fix $\varepsilon > 0$ sufficiently small and note that, in the present situation (if $a, b \in \mathbb{R}$), we can just take $U = [a+\varepsilon, b-\varepsilon] \times \mathbb{R}^n$ and L independently from $(t_0, x_0) \in [a+\varepsilon, b-\varepsilon] \times \mathbb{R}^n$[38]. In this way, once we have fixed $(t_0, x_0) \in U$, in order to remain inside U, $\overline{\delta}_1$ has only to satisfy, independently from x_0,

$$\overline{\delta}_1(t_0) = \min\{t_0 - a - \varepsilon, b - \varepsilon - t_0\},$$

and δ_2 can be any positive value. Moreover, let us not that, by (2.11), there exists a constant $c > 0$ (depending only on ε) such that

$$\|f(t, x)\| \leq L\|x\| + c \quad \forall (t, x) \in [a + \varepsilon, b - \varepsilon] \times \mathbb{R}^n. \tag{2.12}$$

Indeed, we have

$$\begin{aligned} &\|f(t, x)\| = \|f(t, x) - f(t, 0) + f(t, 0)\| \leq \|f(t, x) - f(t, 0)\| + \|f(t, 0)\| \\ &\leq L\|x\| + \max_{s \in [a+\varepsilon, b-\varepsilon]} \|f(s, 0)\| = L\|x\| + c. \end{aligned}$$

Hence, for any $\delta_2 > 0$ and for any $(t_0, x_0) \in [a + \varepsilon, b - \varepsilon] \times \overline{B}(x_0, \delta_2)$, we have

[38] If $a = -\infty$ or $b = +\infty$, we have to replace $a + \varepsilon$ with an arbitrary $K < 0$ and $b - \varepsilon$ with an arbitrary $H > 0$, respectively.

$$\max_{(t,x)\in[t_0-\overline{\delta}_1(t_0),t_0+\overline{\delta}_1(t_0)]\times\overline{B}(x_0,\delta_2)} \|f(t,x)\| \leq L\delta_2 + L\|x_0\| + c.$$

We can then take $M = L\delta_2 + L\|x_0\| + c$, which depends on x_0 and on ε (via c), but not on t_0. Hence, if we take

$$\delta_2 = L\|x_0\| + c,$$

which is independent from t_0, we then get

$$\frac{\delta_2}{M} = \frac{1}{1+L},$$

which is definitely independent from $(t_0, x_0) \in [a+\varepsilon, b-\varepsilon] \times \mathbb{R}^n$ and from ε. In this way we finally get

$$0 < \delta_1(t_0) \leq \min\left\{t_0 - a - \varepsilon, b - \varepsilon - t_0, \frac{1}{1+L}\right\},$$

depending on $t_0 \in]a+\varepsilon, b-\varepsilon[$, such that, for every initial value $(t_0, x_0) \in]a+\varepsilon, b-\varepsilon[\times\mathbb{R}^n$, the unique local solution (uniqueness in the sense of Definition 2.8) exists in the interval $]t_0 - \delta_1(t_0), t_0 + \delta_1(t_0)[$. Hence, for $0 < \varepsilon \leq \frac{1}{1+L}$, and for all initial value $(t_0, x_0) \in]a+2\varepsilon, b-2\varepsilon[\times\mathbb{R}^n$ we can take constantly

$$\delta_1 = \varepsilon.$$

This means that, for all possible initial value $(t_0, x_0) \in]a+2\varepsilon, b-2\varepsilon[\times\mathbb{R}^n$, the unique local solution exists, at least, in the whole interval $]t_0 - \varepsilon, t_0 + \varepsilon[$.

Now, let us definitely fix the initial value $(t_0, x_0) \in]a+2\varepsilon, b-2\varepsilon[\times\mathbb{R}^n$, and take

$$\begin{array}{l} t_1 \in]t_0, t_0+\varepsilon[,\ t_2 \in]t_1, t_1+\varepsilon[,\ \ldots,\ t_k \in]t_{k-1}, t_{k-1}+\varepsilon[,\ t_{k+1} \in]t_k, t_k+\varepsilon[\\ t^1 \in]t_0-\varepsilon, t_0[,\ t^2 \in]t^1-\varepsilon, t^1[,\ \ldots,\ t^h \in]t^{h-1}-\varepsilon, t^{h-1}[,\ t^{h+1} \in]t^h-\varepsilon, t^h[\end{array}$$

It is evident that we can choose $k, h \in \mathbb{N}$ and $t_1, \ldots, t_k, t_{k+1}, t^1, \ldots, t^h, t^{h+1}$ such that $t_1, \ldots, t_k \in]t_0, b-2\varepsilon[$, $t_{k+1} \geq b - 2\varepsilon$, and that $t^1, \ldots, t^h \in]a+2\varepsilon, t_0[$, $t^{h+1} \leq a + 2\varepsilon$.

Let us denote $I_i =]t_i - \varepsilon, t_i + \varepsilon[$ for all $i = 0, \ldots, k$, and $I^j =]t^j - \varepsilon, t^j + \varepsilon[$ for all $j = 1, \ldots, h$.

Hence we define the function $y :]a+2\varepsilon, b-2\varepsilon[\to \mathbb{R}^n$ as

$$y(t) = \begin{cases} y_0(t) & t \in I_0 \quad y_0 \text{ is the unique solution with datum } (t_0, x_0) \\ y_i(t) & t \in I_i \quad y_i \text{ is the unique solution with datum } (t_i, y_{i-1}(t_i)) \quad i = 1, \ldots, k \\ y^j(t) & t \in I_j \quad y^j \text{ is the unique solution with datum } (t^j, y^{j-1}(t^j)) \quad j = 1, \ldots, h. \end{cases}$$

By the local uniqueness on every intervals I_i and I^j, y is well defined and it turns out to be the unique global solution in $]a+2\varepsilon, b-2\varepsilon[$. By the arbitrariness of $\varepsilon > 0$, we then conclude. $\square$

Remark 2.16 *If in Theorem 2.15 we change the global uniform Lipschitz condition (2.11) in the weaker condition*

$$\begin{array}{l} \forall\ \delta > 0 \text{ sufficiently small } \exists\ L_\delta > 0 \text{ such that} \\ \|f(t,x) - f(t,y)\| \leq L_\delta \|x - y\|\ \forall\ (t,x),(t,y) \in]a+\delta, b-\delta[\times\mathbb{R}^n, \end{array} \tag{2.13}$$

then, the same thesis holds true. Indeed, it is sufficient to apply Theorem 2.15 to $]a+\delta, b-\delta[\times\mathbb{R}^n$ *for all* $\delta > 0$*, and then to construct a unique solution in* $]a,b[$.

Exercise. Show that the global existence and uniqueness defined by Definition 2.12 is equivalent to the following

i) a local solution exists and

ii) if $y_1 : I_1 \to \mathbb{R}^n$ and $y_2 : I_2 \to \mathbb{R}^n$ are two solutions on I_1 and I_2 respectively, then $y_1 = y_2$ on $I_1 \cap I_2$.

Exercise. Show, with a counterexample, that the following "natural" conditions do not imply the uniqueness of the local solution in the sense of Definition 2.8:

i) a local solution exists and

ii) if $y_1 : I_1 \to \mathbb{R}^n$ and $y_2 : I_2 \to \mathbb{R}^n$ are two solutions on I_1 and I_2 respectively, then there exists an open interval $J \subseteq I_1 \cap I_2$, containing t_0, such that $y_1 = y_2$ on J.

(*Hint.* Consider, for $t, x \geq 0$ the dynamics[39]

$$f(t,x) = \begin{cases} x^{\frac{2}{3}} & \text{if } 0 \leq x \leq t^4, \\ 0 & \text{if } x \geq t^4, \end{cases}$$

and the corresponding Cauchy problem with $y(0) = 0$.)

Exercise. Show, with a counterexample, that the uniqueness of a solution in an open interval I does not imply the uniqueness of the solution in every open subinterval $J \subseteq I$.

(*Hint.* For instance consider the function f such that, for $(t,x) \in [0,+\infty[\times[0,+\infty[$, is given by

$$f(t,x) = \begin{cases} x^{\frac{2}{3}} & \text{if } 0 \leq x \leq \sqrt{t}, \\ x^{\frac{4x+2t-6\sqrt{t}+2}{3(t+1-\sqrt{t})}} & \text{if } \sqrt{t} \leq x \leq t+1, \\ x^2 & \text{if } x \geq t+1. \end{cases}$$

Consider the Cauchy problem with initial datum $y(0) = 0$, and show that in $[0,+\infty[$ there is only the null solution $y \equiv 0$ but that on every proper subinterval $[0,k[$ there are infinitely many solutions.

Hint of the hint. Note that, apart from the zero solution, every solution starts as one of the solutions in the Subsection 2.3.4 and, outside of an "intermediate linking region", it goes on as the solution in Example 2.4, which does not exist for all times.)

[39] Such a dynamics is discontinuous, but for our purpose here is not a problem.