SUGGESTED EXERCISES FOR THE DOCTORAL COURSE "GAME THEORY"

Exercise 2 1) Consider a non-cooperative game with two players. Let NE, SE_1 and PO respectively denote a Nash Equilibrium, Stackelberg Equilibrium Leader 1, Pareto Optimality. Prove or confute the following implications

$$\begin{array}{l} NE \Longrightarrow SE_1,\ NE \Longrightarrow PO,\ SE_1 \Longrightarrow PO, \\ SE_1 \Longrightarrow\ NE,\ PO \Longrightarrow\ NE,\ PO \Longrightarrow SE_1. \end{array}$$

2) Consider a non-cooperative game with three players. Give a possible definition for everyone of the following Stackelberg-type equilibria[1]: a) SE_{12}: 1 is leader of 2 and 3, 2 is leader of 3 and 1 knows that; b) $SE_{1^2_3}$: 1 is leader of 2 and 3, there is a leader between 2 and 3, 1 knows that but he does not know who is it; SE_{1*}: 1 is leader of 2 and 3 and he knows that there is not a leader between 2 and 3. For every case, if possible, give a formula for the choice of 1. Provide some examples.

Exercise 3 Let us given a cooperative TU game $< N, v >$. A map $\lambda : 2^N \setminus \emptyset \to [0,1],\ S \mapsto \lambda_S$ is said to be <u>a balanced map</u> if

$$\sum_{S | i \in S} \lambda_S = 1 \ \ \forall\, i \in N.$$

The game is said to be <u>balanced</u> if: for every every balanced map λ, the following holds

$$\sum_{S \in 2^N \setminus \emptyset} \lambda_S v(S) \leq v(N).$$

Prove the Bondareva-Shapley theorem: the core of a cooperative game is non-empty if and only if the game is balanced.

Hint. The necessity was already proved in class. For the sufficiency, move along the following steps.

1) Recall the following fundamental duality theorem in linear programming: given the maximization problem (here all the dimensions of vectors and matrices are the appropriate ones, ^{T} means transposition, and the inequalities must be considered as satisfied component by component)

$$(P) \quad \begin{array}{l} \max x^T c \\ \text{subject to } x^T A \leq b^T,\ x \geq 0, \end{array}$$

and its *dual problem*

$$(D) \quad \begin{array}{l} \min b^T y \\ \text{subject to } Ay \geq c,\ y \geq 0, \end{array}$$

[1] The notations are just invented for this exercise.

then, (P) has an optimal solution (i.e. a solution) $\overline{x}$ if and only if (D) has an optimal solution $\overline{y}$ and, in such a case,

$$\overline{x}^T c = b^T \overline{y}.$$

2) Now, suppose the game balanced. Also suppose that $v(S) \geq 0$ for all S (if v may take negative values, then point 2iv) is more delicate).

2i) Consider the following problem (in the unknowns λ_S)

$$\begin{cases} \max \displaystyle\sum_{S \in 2^N \setminus \emptyset} \lambda_S v(S) \\ \text{subject to } \displaystyle\sum_{S, i \in S} \lambda_S \leq 1 \ \forall i \in N, \ \lambda_S \geq 0 \ \forall S, \end{cases}$$

and write it in the form as (P), for suitable vectors c, b and matrix A (you also have to use the vectors $e^S \in \mathbb{R}^n$, where n is the number of players, with $(e^S)_i = 1$ if $i \in S$, $(e^S)_i = 0$ otherwise.)

2ii) Write the dual problem.

2iii) Prove that the constraints in (P) identify a non-empty compact set, and so (P) has a an optimal solution;

2iv) Prove that there certainly exists an optimal solution of (P) which is a balanced map.

2v) By the duality theorem above and by the fact that the game is balanced, conclude that the solutions of the dual problem are element of the core.

Exercise 4 Let K, K_1 be two subsets of $\mathbb{R}^n$, with $K_1 \subseteq K$. We say that K_1 is a retract of K if there exists a continuous function $\pi : K \to K_1$ such that $\pi(x) = x$ for all $x \in K_1$. Prove the following fact: if K has the property that, for every continuous function $f : K \to K$, f has a fixed point, then the same property also holds for every K_1 retract of K.