7 Autonomous systems

7.1 General results. First integrals

Let us consider an autonomous system

$$y'(t) = f(y(t)), \tag{7.1}$$

where $f : A \to \mathbb{R}^n$, with $A \subseteq \mathbb{R}^n$ open, is locally Lipschitz (and hence we have local existence and uniqueness for every initial state, and in particular existence and uniqueness of the maximal solution).

Definition 7.1 *If $y : I \to \mathbb{R}$ is a solution of (7.1), by trajectory or orbit we mean the image of y, that is the curve in $\mathbb{R}^n$ (the phase space) described by the function/parametrization y.*

The fact that the system is autonomous, that is the dynamics f does not explicitly depend on the time t, permits to study the solutions by studying their orbits in the phase space, and then to get several interesting results.

To simplify notations and proofs, we will often assume the following hypothesis:

$$A = \mathbb{R}^n, \quad \text{all the maximal solutions are defined for all time } t \in \mathbb{R}\ ^{95}. \tag{7.2}$$

Here are some first results that hold because of the autonomy.

Proposition 7.2 *i) If y is a solution of (7.1), and $c \in \mathbb{R}$, then the function*

$$\psi : t \mapsto y(t + c)$$

is still a solution.

ii) If y_1 and y_2 are two solutions such that, for some $t_1, t_2 \in \mathbb{R}$ it is

$$y_1(t_1) = y_2(t_2),$$

then we must have

$$y_1(t) = y_2(t + t_2 - t_1)\ \forall\ t \in \mathbb{R}.$$

Proof. i) Just deriving

$$\psi'(t) = y'(t + c) = f(y(t + c)) = f(\psi(t)).$$

ii) For the first point i), $\overline{y}_2(t) = y_2(t + t_2 - t_1)$ is a solution, and in particular $\overline{y}_2(t_1) = y_2(t_2) = y_1(t_1)$. Then, by uniqueness, $\overline{y}_2 = y_1$. □

[95] However, we will later see some quite easy controllable properties which guarantee the existence for all time.

Remark 7.3 *From Proposition 7.2, we have that any orbit corresponds to a one-parameter family of solutions $y(t) = y(t+c)$: if a solution describes an orbit, then every its translation in time describes the same orbit, moreover if two solutions describe the same orbits then they must be the same solution translated in time. Also, two orbits cannot intersect each other: the orbits give a partition of the phase space.*

Remark 7.4 *Let us note that every non autonomous system*

$$y'(t) = f(t, y(t))$$

can be written as an autonomous one just adding the fictitious state-variable t. Indeed, writing $\tilde{y} = (t, y)$ and $\tilde{f}(\tilde{y}) = (1, f(\tilde{y}))$, we get

$$\tilde{y}' = \tilde{f}(\tilde{y}).$$

Of course, in doing that, we have paid the fact that we passed to the larger dimension $n + 1$.

Definition 7.5 *A point $x \in \mathbb{R}^n$ is said an <u>equilibrium point</u> (or a <u>critic/singular point</u>) of the system if $f(x) = 0$. It is obvious that, if x is an equilibrium point, then the set $\{x\} \subset \mathbb{R}^n$ is an orbit, since the function $y(t) \equiv x$ is a solution. Such an orbit is sometimes called a <u>stationary orbit.</u>*

Proposition 7.6 *If $x \in \mathbb{R}^n$ is the limit of a solution when $t \to \pm\infty$, then x is an equilibrium point. Moreover, a non stationary solution y cannot pass through an equilibrium point.*

Proof. The second assertion is obvious, since an equilibrium point is an orbit. Let us prove the first one, for $t \to +\infty$. By absurd, let us suppose that $f(x) \neq 0$. Then there exists a unit versor $\nu \in \mathbb{R}^n$ such that $f(x) \cdot \nu > 0$. Let us take a small ball around x, B, such that, by the continuity of f, for a suitable fixed $\varepsilon > 0$,

$$y \in B \implies f(y) \cdot \nu > \varepsilon > 0.$$

Since by hypothesis of convergence $y(t) \in B$ definitely for $t \geq \overline{t}$ (because $y(t) \to x$ as $t \to +\infty$), for a suitable $\overline{t}$, we have, for all $t \geq \overline{t}$, and integrating in $[\overline{t}, t]$,

$$y'(t) \cdot \nu = f(y(t)) \cdot \nu > \varepsilon \implies y(t) \cdot \nu \geq y(\overline{t}) \cdot \nu + \varepsilon(t - \overline{t}) \to +\infty \text{ as } t \to +\infty,$$

which is an absurd since $y(t) \cdot \nu \to x \cdot \nu \in \mathbb{R}$. [96] $\square$

[96] Roughly speaking: when $y(t) \in B$, by the absurd hypothesis, the trajectory has a scalar velocity which, with respect to the direction ν, is not less than $\varepsilon > 0$; $y(t) \in B$ for all time $t \geq \overline{t}$; these two facts imply that $y(t)$ must exit from the bounded set B. Contradiction.

Definition 7.7 *A solution y is said to be* <u>*periodic*</u> *if there exists a time $T > 0$ (the* <u>*period*</u>*) such that*

$$y(t+T) = y(t) \quad \forall t, \quad y(t+s) \neq y(t) \quad \forall s \in]0, T[.$$

Note that, by this definition, the constant trajectories (i.e. the equilibrium points) are not periodic.

Proposition 7.8 *For an autonomous system with uniqueness, there are only three types of orbits: singular equilibrium points, simple closed curves (i.e* <u>*cycles*</u> *without transversal/tangential self-intersections) which are periodic, and simple open curves (i.e open curves without transversal/tangential self-intersections).*

Proof. The fact that an orbit cannot have a self-intersection is obvious by uniqueness. It is also obvious that the orbit of a periodic solution is a cycle. Hence, we have only to prove that if a non constant solution y is such that $y(t_1) = y(t_2)$ for some $t_2 > t_1$, then it is periodic[97].

Let us define $\delta = t_2 - t_1$. We guess that $y(t+\delta) = y(t)$ for all t. Indeed, the function $\psi : t \mapsto y(t+\delta)$ is still a solution and it satisfies $\psi(t_1) = y(t_2) = y(t_1)$. Hence it coincides with y and the guess is proved. We obtain the periodicity of y, in the sense of Definition 7.7 if we prove that

$$T = \inf \left\{ \tau > 0 \middle| y(t+\tau) = y(t) \ \forall t \right\} > 0.$$

Let $\mathcal{P}$ be the set whose infimum we are going to consider. Note that $\mathcal{P}$ is not empty since $\delta \in \mathcal{P}$. Moreover, let us note that if $\tau \in \mathcal{P}$ then $m\tau \in \mathcal{P}$ for all positive integers m. Also, since y is continuous, $\mathcal{P}$ is closed in $]0, +\infty[$, that is if $\tau > 0$, $\tau_n \to \tau$ and $\{\tau_n\}_n \subset \mathcal{P}$, then $\tau \in \mathcal{P}$ (i.e. every strictly positive accumulation point of $\mathcal{P}$ belongs to $\mathcal{P}$ itself). By absurd, let us suppose that $T = 0$. Then, for every $\varepsilon > 0$ there exists $\tau \in \mathcal{P}$ such that $0 < \tau < \varepsilon$. Moreover, fixed such a τ, for any real number $c > 0$, for the archimedean property of $\mathbb{R}$, we find $m \in \mathbb{N} \setminus \{0\}$ such that

$$(m-1)\tau \leq c \leq m\tau \implies 0 \leq m\tau - c \leq \tau \leq \varepsilon,$$

which means that $c > 0$ is an accumulation point of $\mathcal{P}$ and so that $c \in \mathcal{P}$. For the arbitrariness of $c > 0$, we conclude that y is constant, which is a contradiction. □

Definition 7.9 *A C^1 function $E : \mathbb{R}^n \to \mathbb{R}$ is said a* <u>*first integral*</u> *of the motion for the system if E is constant along any trajectory of the system. This in particular means that the function $t \mapsto E(y(t))$ is constant for any trajectory y, which is equivalent to say that its derivative is zero, that is*

[97] As stated in the beginning of the section, we are supposing that all the solutions are defined for all times. However, it can be easily proved that, if a non constant solution satisfies $y(t_1) = y(t_2)$ for some two different instants t_1, t_2 of its interval of definition, then it is prolongable for all times (and also periodic).

$$\nabla E(y(t)) \cdot y'(t) = 0 \implies \nabla E(y(t)) \cdot f(y(t)) = 0 \ \forall y(\cdot).$$

Since the trajectories are a partition of the phase space $\mathbb{R}^n$, we can equivalently say that E is a first integral if and only if

$$\nabla E(x) \cdot f(x) = 0 \quad \forall \ x \in \mathbb{R}^n.$$

Remark 7.10 *If E is a first integral, then every orbit is entirely contained in a level set of E. Moreover, in the general case of a maximal orbit y defined in $\tilde{I}$, if y is contained in a bounded level set of E, then $\tilde{I} =]-\infty, +\infty[$. Indeed, in that case the derivatives are bounded. In particular, if E has all the level set bounded, then every solution is prolongable for all times.*

7.2 Bidimensional systems

The particular case of bidimensional system is quite favorable. Indeed the phase-space is the plane $\mathbb{R}^2$, where we can easier draw and analyze the orbits. Moreover, it can be easier to find a possible first integral and also, since the level sets of the first integrals are (generally) curves, the orbits coincide with (at least) pieces of such curves.

Proposition 7.11 *Let us consider the bidimensional system*

$$\begin{cases} x' = F(x, y) \\ y' = G(x, y). \end{cases} \tag{7.3}$$

i) The equilibrium points are the solution of the (nonlinear) algebraic system

$$\begin{cases} F(x, y) = 0 \\ G(x, y) = 0. \end{cases}$$

ii) If $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a potential of the differential 1-form

$$\omega(x, y) = G(x, y)dx - F(x, y)dy,$$

then φ is a first integral for the system (7.3).

iii) If a level set of a first integral E is a simple closed curve[98] which does not contains equilibrium points for the systems, then it exactly coincides with a periodic orbit.

Proof. The point i) is obvious by definition.

For the point ii), let $(x(\cdot), y(\cdot))$ be a solution, then we have

$$\begin{array}{l} \dfrac{d}{dt}\varphi(x(t), y(t)) = G(x(t), y(t))x'(t) - F(x(t), y(t))y'(t) = \\ G(x(t), y(t))F(x(t), y(t)) - F(x(t), y(t))G(x(t), y(t)) = 0. \end{array}$$

[98]Note that, in this setting, “closed curve” also implies that it is bounded and of finite length.

iii) Since the curve is bounded, then the solution is defined for all times. Since its scalar velocity $\|f(y(t))\|$ is uniformly greater then zero (there are no equilibrium points and the curve is compact), then the trajectory must pass two times through the same point. Hence, it is periodic. $\square$

Remark 7.12 *If the differential form $\omega = Gdx - Fdy$ is not exact, but it has an integrand factor $\lambda(x, y) > 0$, then a potential E of the form $\lambda\omega$ is still a first integral of the system.*

7.2.1 Qualitative studies (III)

Here we sketch a list of points which may be addressed when studying the qualitative behavior of the orbits for a bidimensional autonomous system.

i) Find the possible equilibrium points.

ii) Find a possible first integral (for instance searching for a potential of the associated differential form).

iii) If a first integral E exists, then study E: stationary points, relative maximum points, relative minimum points, saddle points, absolute maxima and minima... This may permit to understand the behavior of the level curves, which are the projections on $\mathbb{R}^2$ of the intersections in $\mathbb{R}^3$ between the graph of E and the horizontal planes. Another way may be directly study the level curves in $\mathbb{R}^2$ via their implicit formulations $E(x, y) = c$, at least when such equation is (easily) invertible with respect to x or to y. Finally, also some properties of E as convexity and coercivity[99] may be useful.

iv) Recall that: orbits may not intersect each other, the orbits form a partition of the phase space. Moreover if a level curve is a closed curve that does not contain equilibrium points, then it coincides with a cycle (a periodic orbit).

v) Check, if possible, whether the solutions are defined for all times or not. This can be done, for instance, looking to the boundedness of the level curves of E.

vi) Note that a closed level curve of E (and hence a cycle) must moves around a stationary point of E[100].

vii) Study some suitable level curves of E. For instance the ones passing through the equilibrium points, or the zero-level curves, which may be easier to be studied.

viii) Find the versus of moving along the orbits. This can be done by studying the sign of F and G respectively. Also note that, by the continuity of F and G, such a versus is "continuous", since it is the versus of the tangent vector (F, G). Hence, we cannot approximate an orbit with other orbits moving in opposite direction.

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[99] "Coercivity" means that $\lim |E(x)| = +\infty$ when $\|x\| \to +\infty$, or more generally, when x approximates the boundary of the domain of E.

[100] Again, by "closed curve" we also mean that it is bounded, and hence it is compact (since, being a cycle, it is certainly "topologically closed"). Since it is a level curve of E, that is E is constant on it, then, by a simple generalization of the one-dimensional Rolle theorem, there must be a stationary point of E in the region inside the curve. Also note that the stationary points of E are strictly related to the equilibrium points (they almost always coincide).

Example 7.13 Let us recall the Lotka-Volterra system

$$\begin{cases} x' = (\alpha - \beta y)x \\ y' = (-\gamma + \delta x)y, \end{cases}$$

where $\alpha, \beta, \gamma, \delta > 0$ are fixed, and we are looking for solutions $(x(t), y(t))$ in the first quadrant only, that is $x(t), y(t) > 0$.

The only equilibrium point is $(x_0, y_0) = (\gamma/\delta, \alpha/\beta)$, and a first integral is[101]

$$E(x, y) = -\gamma \log x + \delta x - \alpha \log y + \beta y.$$

The study of E gives that: (x_0, y_0) is the only stationary point of E and it is the absolute minimum, E is strictly convex[102], E is coercive, that is it tends to $+\infty$ when x or y tends to 0 (i.e. when the point (x, y) tends to the axes.), and also when $x, y \to +\infty$. Hence, its level curves are closed curves around (x_0, y_0).

We easily conclude that the orbits are periodic (cycles), they are defined for all times and that they counterclockwise move around the equilibrium point.

7.2.2 Some exercises

1) Let us consider the second order autonomous scalar equation

$$y'' = f(y).$$

As usual we can transform it in a first order autonomous bidimensional system

$$\begin{cases} y_1' = y_2 \\ y_2' = f(y_1). \end{cases}$$

Prove that, if F is a primitive of f, then

$$E(y_1, y_2) = \frac{y_2^2}{2} - F(y_1)$$

is a first integral for the system[103].

2) For the following systems/second order equations, plot a qualitative picture of the orbits, and check, if possible, whether the solutions are defined for all times, and whether the equilibrium points are stable, asymptotically stable or unstable[104].

[101] It can be found using the integrand factor $1/(xy)$.

[102] Its Hessian matrix is everywhere positively definite.

[103] In a mechanical point of view, y_1 is the position and y_2 is the velocity. Hence E is the "total energy" of the system: kinetic energy plus potential energy. Since the trajectories move along the level curves of E, that is E is constant along the trajectories, then the system is conservative: the total energy is kept constant.

[104] For the concept of stability see next paragraph.

2i)

$$\begin{cases} x' = (3-x)(x+2y-6) \\ y' = (y-3)(2x+y-6). \end{cases}$$

2ii)

$$\begin{cases} x' = 3y^2 \\ y' = 3x^2. \end{cases}$$

2iii)

$$y'' = y^2 + 2y.$$

2iv) (pendulum without friction)

$$y'' = -k \sin y, \quad k > 0.$$

Some words. The equation is the model for the oscillations of a point of mass m hanged to an extremum of a rigid rod which has negligible mass, length equal to ℓ and is free to rotate in a vertical plane around its other (fixed) extremum, only subject to the gravity force. Denoting by φ the radiant angle of the rod with respect to the downward position, the Newton equation of the motion is

$$m\ell\varphi''(t) = -mg\sin(\varphi(t)),$$

which corresponds to our equation with $y = \varphi$ and $k = g/\ell$.

Following the first exercise of this section, we write our equation as the autonomous bidimensional system ($z_1 = y$ angle, $z_2 = y'$ angular velocity)

$$\begin{cases} z_1' = z_2 \\ z_2' = -k \sin z_1, \end{cases}$$

which, by periodicity, may be studied only for $z_1 \in [-\pi, \pi]$, and hence the equilibrium points are $(-\pi, 0), (0, 0), (\pi, 0)$. A first integral is

$$E(z_1, z_2) = \frac{1}{2}z_2^2 + (k - k\cos z_1) \geq 0,$$

and hence the level curves are the curves of equations

$$z_2 = \pm\sqrt{2(c - k + k\cos z_1)}, \quad c \geq 0.$$

Analyzing all the case for $c \geq 0$ we get that there exist: 1) cycles around the equilibrium point $(0, 0)$, 2) heteroclinic[105] orbits connecting the equilibrium points $(-\pi, 0), (\pi, 0)$, 3) open orbits (not connecting any equilibrium points: they are open in the strip $[-\pi, \pi] \times \mathbb{R}$, but they indeed reply by periodicity in the whole $\mathbb{R}^2$).

2v)

$$y'' = -ye^y.$$

[105] In general, orbits connecting two different equilibrium points are called "heteroclinic", whereas orbits connecting the same equilibrium point are called "homoclinic".

7.3 Stability

To study the stability of the system means to understand how much the trajectories are sensible in (small) change of the initial value: if we little change the initial value, what happens to the trajectory? Does it remain "near" to the initial one or not?

We have already met a "stability result": Proposition 6.12. It says that, on the compact sets of time, if we little change the initial point x, then the trajectories do not "change to much": they uniformly converge to the trajectory starting from x. But that proposition says nothing about what happens when $t \to +\infty$: a stability result for compact set of time does not imply stability for all time, the distance between the trajectories may diverge when $t \to +\infty$.

In this section we assume that all the solutions are defined for $t \in [0, +\infty[$ and we address their behavior when $t \to +\infty$ (a similar analysis may be done when $t \to -\infty$). We are going to use the (flow-) notation $\Phi_0(\cdot, x)$ for the solution starting from x at $t = 0$. For all this section we assume that the autonomous system $y' = f(y)$ satisfies the usual hypothesis for existence and uniqueness.

This section is rather sketched.

Definition 7.14 *Let $x \in \mathbb{R}^n$ be fixed. The solution $\Phi_0(\cdot, x)$ is said:*
i) stable if: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z \in B(x, \delta) \implies \|\Phi_0(t, x) - \Phi_0(t, z)\| \leq \varepsilon \ \forall\ t \geq 0;$$

ii) asymptotically stable if: it is stable and moreover

$$\lim_{t \to +\infty} \|\Phi_0(t, x) - \Phi_0(t, z)\| = 0 \ \forall\ z \in B(x, \delta);$$

iii) unstable in all the other cases.

Remark 7.15 *Point i) of the previous definition says that the trajectory $\Phi_0(\cdot, x)$ is stable if for every "tubular" neighborhood of it, there exists a ball around x such that, starting from any point of such a ball, we remain inside the tube for all $t \geq 0$. Point ii) does not only require the remaining inside the tube, but also requires that we better and better approximate the trajectory $\Phi_0(\cdot, x)$.*

It is interesting, both from a theoretical and applicative point of view, to study the stability of the equilibrium points. In that case, the trajectory is just the point, and hence it is stable if we can remain as a close to it as we want. It is asymptotically stable if we also converge to the equilibrium point.

In the case of asymptotically stable equilibrium point x, we define its basin of attraction as

$$\Omega = \left\{ z \in \mathbb{R}^n \,\middle|\, \lim_{t \to +\infty} \Phi_0(t, z) = x \right\}.$$

We say that x is globally asymptotically stable if $\Omega = \mathbb{R}^n$, that is, whichever the initial point is, we converge to x.

If x is an equilibrium point, that is $f(x) = 0$, then, by the change of variable and dynamics: $z = y - x$ and $g(z) = f(z + x)$, we get the equivalent system $z' = g(z)$ with $\zeta = 0$ as equilibrium point. Hence we may restrict the study to the case where the equilibrium point is the origin.

7.3.1 Stability for linear systems

For the linear (homogeneous) systems $y' = Ay$, the study of the stability of the equilibrium point $x = 0$ is rather easy[106].

Proposition 7.16 *Given the linear homogeneous system $y' = Ay$, the origin is:*

i) globally asymptotically stable if and only if $Re(\lambda) < 0$ for all $\lambda \in \mathbb{C}$ eigenvalues of the matrix A;

ii) stable (but not asymptotically) if and only if $Re(\lambda) \leq 0$, there exists a pure imaginary eigenvalue (i.e. $Re(\lambda) = 0$), and all the pure imaginary eigenvalues has algebraic multiplicity 1;

iii) unstable in all the other cases.

Proof (very sketched). Just arguing as for the linear n-order scalar equation, it can be easily seen that the solutions of the linear system are linear combination of addenda of the following type

$$ht^m e^{Re(\lambda)t}(\cos(Im(\lambda)t) \pm \sin(Im(\lambda)t)),$$

where $h \in \mathbb{R}^n$ is a suitable non null vector. Hence, point i) is almost immediate. For point ii) just observe that we are requiring that, if $Re(\lambda) = 0$, then the multiplicity is 1. This implies that $m = 0$ and hence the addendum is $h(\cos(Im(\lambda)t) \pm \sin(Im(\lambda)t))$ which does not converge to the origin but stays there around. $\square$

7.3.2 On the Liapunov function

For the general case of a nonlinear system $y' = f(y)$, with $f(0) = 0$, the stability of the origin can be studied by the help of a suitable function.

Definition 7.17 *Let $A \subseteq \mathbb{R}^n$ be a neighborhood of the origin. A C^1 function $V : A \to \mathbb{R}$ is said a Liapunov function for the system if*

i) (positively definite) $V(x) \geq 0$ for all $x \in A$ and $V(x) = 0 \iff x = 0$

ii) (decreasing along trajectories) $\nabla V(x) \cdot f(x) \leq 0$ for all $x \in A$.

[106] Note that $x = 0$ is always an equilibrium point for a linear homogeneous system, since $A0 = 0$.

Note that the condition ii) says that, given a trajectory $y(\cdot)$ in A, then the function of time $t \mapsto V(y(t))$ is not increasing.

We do not prove the following theorem.

Theorem 7.18 *If there exists a Liapunov function, then the origin is stable. Moreover, if the Liapunov function is strictly decreasing along the trajectories (i.e. $\nabla V(x) \cdot f(x) < 0$), then the origin is asymptotically stable.*

Remark 7.19 *The Liapunov function is something like a first integral (even if in general is harder to be found). The difference is the following: for the case of the first integral, the trajectories live for all times inside the level set, instead for the Liapunov function, the trajectories tend to leave the level set and to point towards the origin (the lowest value of V). For instance, in the case of strict decreasing along the trajectories, let S_c be the level set of value $c > 0$ for V. Then ∇V is orthogonal to S_c and points towards the higher value of V. Hence the condition $\nabla V \cdot f < 0$ means that the field f is strictly pointing towards the lower value of V and so the trajectory $y(t)$ enters in the region $V < c$. Since this happens at all times, in the limit the trajectory tends to the origin (the lowest value).*

7.3.3 On the linearization method

Another way to study the stability in the nonlinear case is, as it usually happens for nonlinear problems, to make a linearization and try to apply the already known results for the linear case.

Proposition 7.20 *Let the origin be an equilibrium point for $y' = f(y)$, with f of class C^1. We may expand f with the first-order Taylor formula around the origin*[107]

$$f(x) = Df(0)x + o(\|x\|) \quad x \to 0,$$

where $Df(0)$ is the Jacobian matrix of f in 0. Let us consider the linearized system $y' = Df(0)y$, and suppose that $Df(0)$ is not singular. If the origin is asymptotically stable for the linearized system, then it is also asymptotically stable for the nonlinear system. In general, the converse is not true.

7.3.4 On the limit cycles

We know that if a trajectory converges to a point x for $t \to +\infty$, then x is an equilibrium point. However, the trajectories may in general have other behaviors as $t \to +\infty$. For instance they may approximate (or tend to) a cycle given by a periodic orbit. In that case, the behavior of the trajectory is something like a spiral which moves around the cycle and tends to it, without reaching it, of course. In that case we say that such a cycle is a limit cycle.

To check the existence of a limit cycle is of course harder than checking the existence of an equilibrium point. We state the following theorem without proof. It holds for bidimensional systems.

[107] Since the origin is an equilibrium, then $f(0) = 0$.

Theorem 7.21 *(Poincaré-Bendixson). Let us consider a bidimensional autonomous system $y' = f(y)$, with $f \in C^1$. If there exists a bounded open set $\Omega \subseteq \mathbb{R}^2$ such that: every trajectory which enters in it will stay inside Ω for all the other times, and such that it does not contains equilibrium points, then there exists a limit cycle inside Ω.*