5 Prolongation of solutions

In the local existence and uniqueness result for the Cauchy problem, Theorem 2.10, we have established the existence of $\delta > 0$ such that the solution exists unique in the interval $]t_0 - \delta, t_0 + \delta[$. Of course, such a constructed δ is not in general the optimal one[73], that is we can probably obtain existence and uniqueness also in some larger interval I. In other words, we can probably prolong the solution beyond $t_0 + \delta$ or beyond $t_0 - \delta$.

5.1 Maximal solutions and existence up to infinity

Let us consider the first order system of equations in normal form

$$y' = f(t, y), \tag{5.1}$$

where $f : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^{n+1}$ open.

Definition 5.1 *A solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$ of (5.1), is said to be a maximal solution if there is not another solution $y : I \to \mathbb{R}^n$ such that $\tilde{I} \subseteq I$, $\tilde{I} \neq I$, and $y(t) = \tilde{y}(t)$ for all $t \in \tilde{I}$. In other words, the solution $\tilde{y}$ is maximal if it is not prolongable beyond its domain $\tilde{I}$.*

In a similar way we define a maximal solution for a Cauchy problem associated to (5.1)

Remark 5.2 *Let us note the difference of Definition 5.1 with the definition of the global solution, Definition 2.3, and with the definition of the unique global solution, Definition 2.12. In the first case we a priori fix an interval of existence (which we do not make in Definition 5.1), in the second case we require the uniqueness, whereas in Definition 5.1 we are not concerning with uniqueness: a maximal solution may exist without being unique as solution (even locally)[74]. However, as we are going to see, the two notions are strongly related.*

If the Cauchy problem associated to (5.1) has local existence and uniqueness for all initial data $(t_0, y_0) \in A$, then a maximal solution for the Cauchy problem exists and coincides with the unique global solution in Definition 2.12. This the statement of the next theorem, but first we need the following lemma.

Lemma 5.3 *Let us suppose that, for all $(t_0, x_0) \in A$, the Cauchy problem for (5.1) has a unique local solution. Then if two solutions of the equation $y' = f(t, y)$, let us say $\varphi : I \to \mathbb{R}^n$, I open, and $\psi : J \to \mathbb{R}^n$, J open, are equal in a point $\bar{t} \in I \cap J$, then they are equal in all their common interval of definition $I \cap J$.*

Proof. We are going to prove that the set

$$C = \left\{ t \in I \cap J \middle| \varphi(t) = \psi(t) \right\},$$

[73] we have only required that it was sufficiently small in order to be sure that our argumentation holds
[74] Recall, for instance, the example in Paragraph 2.3.4

is a nonempty open-closed subset of $I \cap J$ (for the induced topology)[75]. Hence, since $I \cap J$ is an interval and so connected, we must have $C = I \cap J$ which will conclude the proof.

First of all note that $C \neq \emptyset$ since $\bar{t} \in C$ by definition.

Let $t_n \in C$ converge to $t^* \in I \cap J$. Then, by definition of C, $\varphi(t_n) = \psi(t_n)$ for all n. Since both φ and ψ are continuous on $I \cap J$ (they are solutions), we get $\varphi(t^*) = \psi(t^*)$, and so $t^* \in C$, which turns out to be closed.

Let us now take any $t^* \in C$ and consider the Cauchy problem with datum (t^*, y^*) where $y^* = \varphi(t^*) = \psi(t^*)$. By hypothesis, such a problem has a unique local solution. Since both φ and ψ are solution of such a Cauchy problem, they must coincide in an interval $]t^* - \delta, t^* + \delta[$, which is then contained in C, which turns out to be open. $\square$

Theorem 5.4 *Let us suppose that, for all $(t_0, x_0) \in A$, the Cauchy problem for (5.1) has a unique local solution. Then for any initial datum, the Cauchy problem has a unique maximal solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$, which also turns out to be the unique global solution in the sense of Definition 2.12. The interval $\tilde{I}$ is said the maximal interval of existence.*

Proof. For a fixed initial datum, let us consider the general integral of the Cauchy problem, and define $\tilde{I}$ as the union of all intervals I such that there exists $y_I : I \to \mathbb{R}^n$ solution of the Cauchy problem.

First of all note that $\tilde{I}$ is an open interval containing t_0, since so are all the intervals I. Now we define

$$\tilde{y} : \tilde{I} \to \mathbb{R}^n, \quad y \mapsto y_I(t) \ \text{ if } t \in I. \tag{5.2}$$

Using Lemma 5.3 it is easy to see that (5.2) is a good definition[76], and also that $\tilde{y}$ is the unique maximal solution, as requested. $\square$

Remark 5.5 *If A is a strip: $]a, b[\times\mathbb{R}^n$, then the condition*

$$\exists\, c_1, c_2 \geq 0 \text{ such that } \|f(t,x)\| \leq c_1\|x\| + c_2 \ \forall (t,x) \in A, \tag{5.3}$$

together with the usual hypotheses of continuity and Lipschitz continuity (2.7), guarantees the existence of the maximal solution $\tilde{y}$ in the whole interval $]a, b[$[77]. Indeed, this is exactly what we have proven in the global existence and uniqueness Theorem 2.15, where

[75] Since $I \cap J$ is open, C is open in $I \cap J$ for the induced topology if an only if it is open in $\mathbb{R}$, and it is closed for the induced topology if and only if, whenever $t_n \to t^*$ as $n \to +\infty$, with $t_n \in C, t^* \in I \cap J$, then also $t^* \in C$.

[76] If $t \in I \cap J$ then $y_I(t) = y_J(t)$.

[77] Condition (5.3) says that if f, that is the derivative of $\tilde{y}$, has a linear behavior as $\|\tilde{y}\| \to +\infty$, then the solution $\tilde{y}$ exists for all times $]a, b[$. As example, think to the scalar case with $\tilde{y}, \tilde{y}' \geq 0$; hence we have $0 \leq \tilde{y}' \leq c_1\tilde{y} + c_2$, and it is believable that $\tilde{y}$ must stay under the solution of $y' = c_1 y' + c_2$ which is an exponential function. Since the exponential function exists for all times, that is it is finite for all times, then $\tilde{y}$ cannot go to $+\infty$ before the time b, and hence it must exist until b. In the next section we will formalize the comparison between solutions.

we used (2.12) which is nothing but (5.3), which, in that case, was given by the global uniform Lipschitz condition (2.11). Another simple condition which implies (5.3) is the boundedness of f.

Moreover if f,besides satisfying (5.3), is also defined and continuous in $[a,b[\times\mathbb{R}^n$ or $]a,b]\times\mathbb{R}^n$ (which implies a or b finite), then the limits $\lim_{t\to a^+}\tilde{y}(t)$ or $\lim_{t\to b^-}\tilde{y}(t)$ exist in $\mathbb{R}^n$, that is, the solution is prolongable till $[a,b[$ or $]a,b]$[78]*. This is easily seen just slightly modifying the proof of Theorem 2.15.*

Note that, if, for instance, $A=]a,+\infty[\times\mathbb{R}^n$, and f satisfies (2.7) and (5.3) then, by Remark 5.5, the maximal interval of existence is $]a,+\infty[$, that is the maximal solution exists for all times $t\to+\infty$. It similarly happens if $A=]-\infty,b[\times\mathbb{R}^n$.

In Remark 5.5, we gave some results about the behavior of the maximal solution on a strip when the extrema of the maximal interval are approached. Here we want to say something similar for the general case of A not a strip and in the case of strip but with a different condition than (5.3).

Proposition 5.6 *Let us suppose that $f:A\to\mathbb{R}^n$, with $A\subseteq\mathbb{R}^{n+1}$ open, satisfies the usual conditions for local existence and uniqueness of the Cauchy problem (see Theorem 2.10). Let $\tilde{y}:\tilde{I}\to\mathbb{R}^n$ be a maximal solution of the equation $y'=f(t,y)$, and also suppose that, defined $\beta=\sup\tilde{I}$, there exists $c\in\tilde{I}$ such that $\tilde{y}'$ is bounded in $[c,\beta[$. Then, we have the following alternative:*

i) $\beta=+\infty$

otherwise

ii) the limit $\lim_{t\to\beta^+}\tilde{y}(t)=x_\beta$ exists in $\mathbb{R}^n$, but $(\beta,x_\beta)\notin A$.

A similar conclusion holds for $\alpha=\inf\tilde{I}$.

Proof. By our hypothesis, $\tilde{y}$ is Lipschitz (and in particular uniformly continuous) in $[c,\beta[$, since its derivative is bounded. Hence, if $\beta<+\infty$, we can prolong $\tilde{y}$ up to the boundary β, that is the limit $x_\beta\in\mathbb{R}^n$ exists. However, if by absurd $(\beta,x_\beta)\in A$, which is open, then $\tilde{y}'_-(\beta)=f(\beta,x_\beta)$[79] and, again by hypothesis, we can extend the solution beyond β. This is a contradiction since $\tilde{y}$ is maximal. □

Theorem 5.7 *Let f be as in Proposition 5.6, let $\tilde{y}:\tilde{I}\to\mathbb{R}^n$ be a maximal solution of $y'=f(t,y)$, and let $K\subset A$ be a compact set. Then there exists $[a,b]\subset\subset\tilde{I}$*[80] *such that*

$$(t,\tilde{y}(t))\notin K\quad\forall\, t\in\tilde{I}\setminus[a,b].$$

We then say that the maximal solutions definitely exit from any compact set in the domain of f.

[78] This means, for instance, that $\lim_{t\to b^-}\tilde{y}(t)=x_b$ exists in $\mathbb{R}^n$ and that $\tilde{y}'_-(b)=f(b,x_b)$, where $\tilde{y}'_-$ is the left-derivative. The validity of the last formula is easily seen using the integral representation.

[79] See Footnote 78. Such an equality guarantees that the extension beyond β is really an extension of $\tilde{y}$, since it glues to it in $t=\beta$ in a C^1-manner.

[80] This means that $\inf\tilde{I}<a\le b<\sup\tilde{I}$.

Proof. Let us define $\beta = \sup \tilde{I}$, and by absurd, let us suppose that there exists a sequence $t_n \in \tilde{I}$ converging to β such that $(t_n, \tilde{y}(t_n)) \in K$ for all n. Hence, possibly extracting a subsequence, there exists $x_\beta \in \mathbb{R}^n$ such that

$$\lim_{n\to+\infty}(t_n, \tilde{y}(t_n)) = (\beta, x_\beta) \in K \subset A.$$

In particular, this means that $\beta < +\infty$. We now get the contradiction since, by virtue of the convergence, there exists a ball $B \subseteq A$ which contains (β, x_β) and $(t_n, \tilde{y}(t_n))$ for sufficiently large $n \geq \overline{n}$. If then we look back to the proof of the local existence and uniqueness Theorem 2.10, we see that there should exist a common $\delta > 0$ such that the unique local solution of any Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_n) = \tilde{y}(t_n), \end{cases}$$

must exist at least until $t_n + \delta$. By uniqueness, such a local solution must be equal to $\tilde{y}$ itself for $t \leq \beta$. This means that, whenever $t_n > \beta - \delta$, we can prolong $\tilde{y}$ beyond β[81]. A contradiction to the maximality of $\tilde{y}$.

The proof for $\alpha = \inf \tilde{I}$ is similar. □

Remark 5.8 *In the particular case of an autonomous system $y' = f(y)$, with $f : A \to \mathbb{R}^n$ locally Lipschitz and $A \subseteq \mathbb{R}^n$ open, if $\tilde{y} : \tilde{I} \to \mathbb{R}^n$, $\tilde{I}$ open, is a maximal solution and $K \subseteq A$ is compact, then we have the following alternative*

i) $\tilde{y}$ definitely exits from K, that is there exist $a, b \in \tilde{I}$ such that

$$\tilde{y}(t) \notin K \ \forall\, t \in \tilde{I} \setminus [a, b],$$

otherwise

ii) $\sup \tilde{I} = +\infty$ *or* $\inf \tilde{I} = -\infty$.

Indeed, let us first note that we can think to the system as $y' = \tilde{f}(t, y)$, where $\tilde{f} : \tilde{A} \to \mathbb{R}^n$ with $\tilde{A} = \mathbb{R} \times A$, $\tilde{f}(t, x) = f(x)$. Hence, if i) is not verified, we have, for instance, $t_n \to \sup \tilde{I}^-$ with $\tilde{y}(t_n) \in K$. But then, if $\sup \tilde{I} \in \mathbb{R}$ we have, for a suitable compact interval J around it and for large n, $(t_n, \tilde{y}(t_n)) \in J \times K$, which is compact in $\tilde{A}$, and this is a contradiction to Theorem 5.7.

Remark 5.9 *Note that both Proposition 5.6 and Theorem 5.7 say that, if $\beta = \sup \tilde{I} < +\infty$ and A is bounded*[82]*, then*

$$\lim_{t\to\beta^-} dist\,((t, \tilde{y}(t), \partial A)) = 0, \tag{5.4}$$

that is the couple $(t, \tilde{y}(t))$ approximates the boundary ∂A when t approximates β. Moreover, Proposition 5.6 also says that, if $\tilde{y}'$ is bounded, then the limit $\lim_{t\to\beta^-} \tilde{y}(t)$ does exist in $\mathbb{R}^n$ (if $\tilde{y}'$ is not bounded, then the solution may oscillate).

[81] See Footnote 79.

[82] We are here supposing the boundedness of A in order to give an immediate meaning to (5.4), without ambiguities for $\|\tilde{y}\| \to +\infty$.

Now we give another criterium for the existence of the solution up to infinity.

Proposition 5.10 *Let f be again as in Proposition 5.6 and also suppose that A is the strip $]a,+\infty[\times\mathbb{R}^n$. Let $\tilde{y}:\tilde{I}\to\mathbb{R}^n$ be a maximal solution of $y'=f(t,y)$ such that, for some $\tau\in\tilde{I}$ and for some $c_1,c_2\geq 0$, we have*[83]

$$\|\tilde{y}(t)\|\leq c_1+c_2(t-\tau)\quad\forall\ t\geq\tau,\ t\in\tilde{I}. \tag{5.5}$$

Then, $\tilde{y}$ is prolongable up to infinity, that is $\beta=\sup\tilde{I}=+\infty$.

A similar result holds for the prolongation up to $-\infty$.

Proof. By absurd, let us suppose that $\beta<+\infty$. Hence, fixed any $B>c_1+c_2(\beta-\tau)$, the compact cylinder

$$K=\left\{(t,x)\in\mathbb{R}^{n+1}\middle|t\in[\tau,\beta],\|x\|\leq B\right\},$$

is contained in A. Hence, by Theorem 5.7, the maximal solution $\tilde{y}$ must definitely exit from K when times approach β. By (5.5) and by our assumption on B we have, for any $t\geq\tau$, $t\in\tilde{I}$,

$$\|\tilde{y}(t)\|\leq c_1+c_2(\beta-\tau)<B,$$

which implies that $\tilde{y}$ must exit from the cylinder K through the "wall" $\{(\beta,x)|\|x\|\leq B\}$. But this means, as before, that we can prolong $\tilde{y}$ beyond $\tilde{I}$, which is a contradiction since $\tilde{I}$ is the maximal interval. □

5.2 Qualitative studies (I)

The results of the previous paragraph may be used to get information about the qualitative behavior of the solutions of scalar equations, even when an explicit formula for the solutions can not be found (or it is hard to be found). In this way it may be possible to draw a "qualitative" picture of the graphs of the solutions. Here is a list of points that are usually convenient to address.

1) Check the local existence and uniqueness, for instance verifying the hypothesis (2.7).

2) Check the prolongability of the solutions, for instance using (5.3).

3) Find the possible stationary solutions $y\equiv y_0$, which correspond to the values $y_0\in\mathbb{R}$ such that $f(t,y_0)=0$ for all t.

4) Find the region of the plane (t,x) where $f>0$ and where $f<0$ respectively, to get information about the monotonicity of the solutions. Also observe that, when passing from one region to another, the solution must have a relative extremum.

5) Observe that by uniqueness the graphs of the solutions cannot intersect each other.

[83] Also (5.5) is a condition of linearity up to infinity as (5.3), but here it is respect to time instead of space, and moreover it is requested to the solution instead of to the dynamics.

6) Study the limit $\lim_{t\to\pm\infty} y(t)$, when reasonable[84] and possible. Here, we can use some known facts as, for instance: if a C^1 function g satisfies $\lim_{t\to+\infty} g(y) = \ell \in \mathbb{R}$ then it cannot happen that $\lim_{t\to+\infty} g'(t) = \pm\infty$[85].

7) Sometimes it may be useful to check whether there are some symmetries in the dynamics f, since this fact may help in the study. For instance, if f has the following symmetry (oddness with respect to the vertical axis $t = 0$)

$$f(t,x) = -f(-t,x) \ \ \forall\ (t,x) \in \mathbb{R}^2,$$

then the behavior of the solutions on the second and third quadrants is specular with respect to the one in the first and fourth quadrants. Indeed, let $y :]a,b[\to \mathbb{R}$ be a solution with $0 \le a \le b$, then the function

$$\psi :]-b,-a[\to \mathbb{R}, \quad t \mapsto y(-t),$$

is also a solution. This can be easily checked. Let us fix $\tau \in]a,b[$ and consider $x_0 = y(\tau)$. Then we have

$$y(t) = x_0 + \int_\tau^t f(s,y(s))ds \ \ \forall\ t \in]a,b[,$$

and hence, for all $t \in]-b,-a[$, via the change of variable $\xi = -s$,

$$\psi(t) = y(-t) = x_0 + \int_\tau^{-t} f(s,y(s))ds =$$
$$x_0 + \int_{-\tau}^t f(-\xi,y(-\xi))(-d\xi) = x_0 + \int_{-\tau}^t f(\xi,\psi(\xi))d\xi,$$

which means that ψ is solution[86].

8) Sometimes it may be useful to study the sign of the second derivative y''. This can be guessed just deriving the equation[87].

Example 5.11 Given the following scalar equation

$$y'(t) = (t^2 - y)\frac{\log(1+y^2)}{1+y^2}, \tag{5.6}$$

discuss existence, uniqueness, maximal interval of existence and draw a qualitative graph of the solutions.

We are going to analyze the above eight points.

[84] That is when the maximal interval is a left- and/or right- half line.

[85] The simple proof is left as exercise.

[86] Another simpler way to check that ψ is a solution is just to calculate its derivative.

[87] Concerning the existence of the second derivative, note that if $f \in C^m$ then any solution y belongs to C^{m+1}. Indeed, for instance, if $f \in C^1$, then, since the solutions are C^1 by definition, $y' = f(t,y) \in C^1$ and hence $y \in C^2$. We can then go on in this way.

1) Here we have

$$f(t,x)=(t^2-x)\frac{\log(1+x^2)}{1+x^2},$$

which is of class C^1 in whole $\mathbb{R}^2$, which is a strip. Hence, there is local existence and uniqueness for the Cauchy problem with any initial datum $(t_0,x_0)\in\mathbb{R}^2$.

2) Since $\log(1+x^2)\le 1+x^2$ for all $x\in\mathbb{R}$, then, for every $a>0$ we have

$$|f(t,x)|\le |x|+a^2 \quad \forall\ (t,x)\in[-a,a]\times\mathbb{R},$$

which, by Remark 5.5, guarantees the existence of the maximal solutions in the whole interval $]-a,a[$. By the arbitrary of $a>0$ we get the existence of the maximal solution for all times $t\in\mathbb{R}$.

3) The only stationary solution is $y(t)\equiv 0$.

4) The sign of f is given by

$$\begin{cases} f(t,x)>0 & \text{if } x<t^2,\\ f(t,x)<0 & \text{if } x>t^2.\end{cases}$$

This means that the solutions are decreasing if $y(t)>t^2$ and increasing if $y(t)<t^2$. Moreover, when the graph of a solution crosses the parabola $y=t^2$ at time $\overline{t}$, then, at that time, the solution has a relative extremum: a maximum if crosses in the second quadrant (where it can only pass from the increasing region to the decreasing region), and a minimum if crosses in the first quadrant[88].

5) Since the null function $y\equiv 0$ is a solution, then all the solutions which sometimes have a strictly positive value (respectively: a strictly negative value) are strictly positive (respectively: strictly negative) for all times $t\in\mathbb{R}$.

6) Let us consider a solution y negative. We must have $\lim_{t\to-\infty}y(t)=-\infty$. Indeed, since the negative solutions are all increasing, the alternative is $-\infty<\lim_{t\infty}y(t)=\ell<0$. But this fact would imply that $\lim_{t\to-\infty}y'(t)\ne-\infty$, which is absurd since

$$\lim_{t\to-\infty}y'(t)=\lim_{t\to-\infty}(t^2-y(t))\frac{\log(1+y(t)^2)}{1+y(t)^2}=\lim_{t\to-\infty}t^2-\ell\frac{\log(1+\ell^2)}{1+\ell^2}=+\infty.$$

In a similar way, we have $\lim_{t\to+\infty}y(t)=0$. Indeed, y is increasing and bounded above (by zero). Hence it must converge to a finite value ℓ. But, whenever $\ell\ne 0$ we get $\lim_{t\to+\infty}y'(t)=+\infty$ which is an absurd.

Now, let us consider a solution y positive. First of all, let us note that y must cross the parabola $y=t^2$ in the first quadrant, passing from the decreasing region to the increasing one, and remaining on the latter for the rest of the times. Hence we have $\lim_{t\to+\infty}y(t)=+\infty$. Indeed y is increasing and hence the alternative is the convergence

[88]Note that, for instance, a solution cannot cross the parabola in the second quadrant passing from the decreasing region to the increasing region, otherwise it should decrease a little bit in the increasing region too. Similarly it happens in the first quadrant.

to a finite value $\ell > 0$, but also in this case, looking to the behavior of the derivative, we would get an absurd. Concerning the behavior in the second quadrant, we note that every solution must cross the parabola in that quadrant too. Indeed, if not, we would have $\lim_{t\to-\infty} y(t) = +\infty$ with $y(t) \geq t^2$ for all $t \leq 0$ and y decreasing. But this implies

$$0 \geq \lim_{t\to-\infty} y'(t) = \lim_{t\to-\infty} (t^2 - y(t)) \frac{\log(1 + y(t)^2)}{1 + y(t)^2} \geq \lim_{t\to-\infty} -y \frac{\log(1 + y(t)^2)}{1 + y(t)^2} = 0,$$

which is an absurd since, if it is true, we would not have $y(t) \geq t^2$ definitely for $t \to -\infty$. Hence, y must definitely belong to the increasing region and, as before, the only possibility is $\lim_{t\to-\infty} y(t) = 0$.

For this example, there are not evident symmetries and also we let drop point 8), since we already have sufficient information in order to draw a qualitative picture of the solutions. The drawing is left as an exercise.