Minimal decompositions and the geometry of finite sets

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QUANTUM PHYSICS AND GEOMETRY

Workshop - Levico Terme

July 5, 2017

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Setting the problem

Tensor rank decomposition

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E.g. elementary tensors

$$T = v_1 \otimes \cdots \otimes v_m \quad v_i \in \mathbb{C}^{n_i}$$

elementary tensors = product tensors.

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-     how far are we from a minimal decomposition.

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$$\text{uniqueness} \implies \text{minimality}$$

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Is the decomposition above minimal?

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Is the decomposition above also minimal as a general decomposition?
Is there a bound for the length of a minimal decomposition, in terms of k?

Geometric tensors

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The result is a *projective space of tensors* $\mathbb{P}(\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_m})$ or $\mathbb{P}(Sym^d(\mathbb{C}^n))$ or $\mathbb{P}(\Lambda^d(\mathbb{C}^n))$, etc.

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ADVANTAGE: the projective space is *compact*, a property that, in Geometry, makes usually things much easier to study.

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Geometric elementary tensors

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In $\mathbb{P}(\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_m}) = \mathbb{P}^r$ elementary tensors are points in the image of the *Segre map*:

$$s := s_{n_1,\ldots,n_m} : \mathbb{P}(\mathbb{C}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{n_m}) \to \mathbb{P}^r.$$

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The Veronese map equals the Segre map restricted to the linear subspace of symmetric tensors.

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In other words, there are points $P_1, \dots, P_k \in \mathbb{P}(\mathbb{C}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{n_m})$ (resp. $P_1, \dots, P_k \in \mathbb{P}(\mathbb{C}^n)$) such that T lies in the span of $s(P_1), \dots, s(P_k)$ (resp. $v(P_1), \dots, v(P_k)$).

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Secant varieties

The sets of points spanned by k points of X are the *strict secant varieties*.

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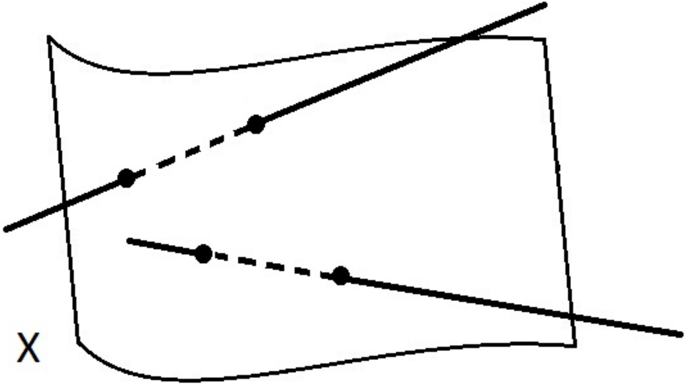
Secant varieties

The sets of points spanned by k points of X are the *strict secant varieties*. Their closures are the *secant varieties* of X

$$\sigma_k(X).$$

Geometric tensors

$$\sigma_2(X)$$



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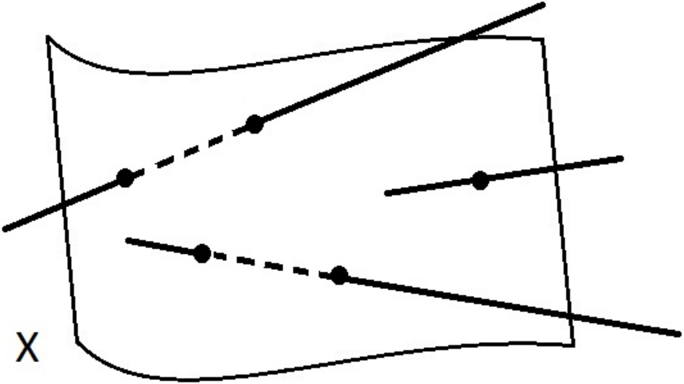
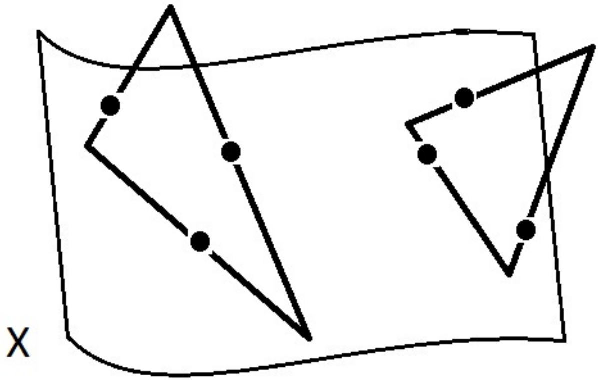


one must include also the limits of secant lines, e.g. tangent lines.

Geometric tensors

$$\sigma_3(X)$$



one must include also the limits.

Kruskal's result for uniqueness

Theorem (Kruskal 1977)

Let $a_1 T_1 + \cdots + a_k T_k$ be a decomposition of $T \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$.
Let r_i be the i-**th Kruskal's rank** of the T_i's. If

$$k \leq \frac{r_1 + r_2 + r_3 - 2}{2}$$

the the decomposition is minimal and unique.

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Each T_j corresponds to a point $P_j \in Y := \mathbb{P}^{n_1 - 1} \times \mathbb{P}^{n_2 - 1} \times \mathbb{P}^{n_3 - 1}$. There are obvious projections $\pi_i : Y \to \mathbb{P}^{n_i - 1}$.

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The *i*-th Kruskal rank r_i corresponds to the maximal integer such that the points $\pi_i(P_j)$'s are in r_i-th *Linear General Position* (LGP),
i.e. any set of cardinality $\leq r_i$ is linearly independent (no three points on a line, no four points on a plane, etc.).

Example: Kruskal's result for uniqueness

Theorem (Kruskal 77)

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(Domanov-DeLathauwer) It can be extended to k-way tensor by repacking them in groups of three.

(LC-Ottaviani-Vannieuwenhoven) It can be extended to symmetric tensor by *reshaping*.

Symmetric reshaping

Theorem (LC-Ottaviani-Vannieuwenhoven SIAM J.MatrixAn. 2017)

Consider tensors in $\mathbb{P}(Sym^d\mathbb{C}^n) \subset \mathbb{P}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})$, where $d_1 + d_2 + d_3 = d$. Then use Kruskal's citerion.

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Most effective for $d_1 = d_2 = \lfloor \frac{1}{2}(d-1) \rfloor$ and $d_3 = d - 2d_1$:

$$k \leq \begin{cases} \frac{3}{2}(n-1) + \frac{1}{2} & \text{if } d = 3, \\ 2(n-1) & \text{if } d = 4, \\ \binom{d_1+n-1}{d_1} + \frac{1}{2}\binom{d_3+n}{d_3} - 1 & \text{if } d \geq 5. \end{cases}$$

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The criterion is *effective*:

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Theorem (Dersken 2013)

Kruskal's range is sharp.

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In geometric terms, the *abstract* (projective) secant variety $A\sigma_4(X)$, X being the image of the 4-veronese map of $\mathbb{P}^3$, has dimension 35, so the map to $\mathbb{P}(Sym^4\mathbb{C}^4) = \mathbb{P}^{34}$ cannot be generically one-to-one (birational).

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Consider symmetric tensors $F \in \mathbb{P}(Sym^4\mathbb{C}^4)$.
F corresponds to a quartic homogeneous polynomial (form) in 4 variables, hence, in geometric terms, to a surface of degree 4 in $\mathbb{P}^3$.

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An exception to the general non-sense principle that tensors of subgeneric rank are identifiable.
Indeed all the exceptions are classified (for symmetric tensors).
(Ballico 2005, LC - Ottaviani - Vannieuwenhoven - TAMS 2016)
(for generic rank Galuppi-Mella 2017)

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(Angelini - Bocci - LC, Lin.Multlin.Alg. 2017)

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So we have a trustable method to determine wether a quartic of rank ≤ 6 is identifiable or not.

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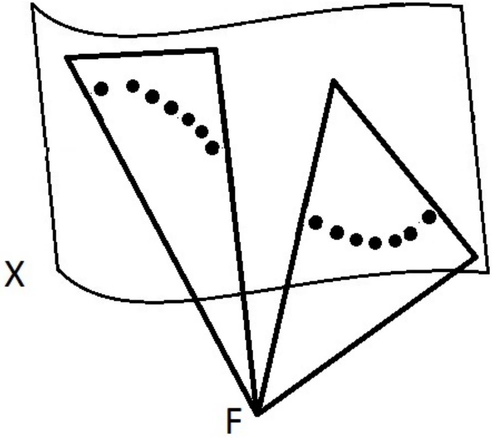
Quartics of rank 7 in 4 variables

F non identifiable of rank 7.



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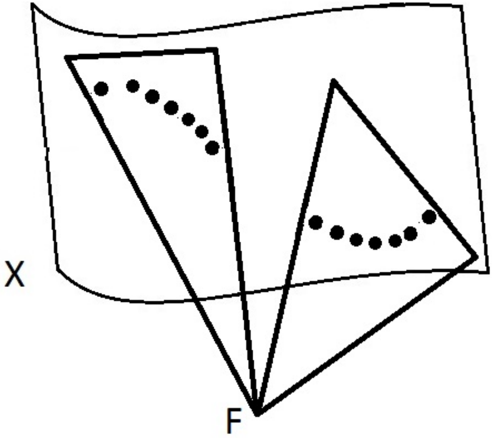

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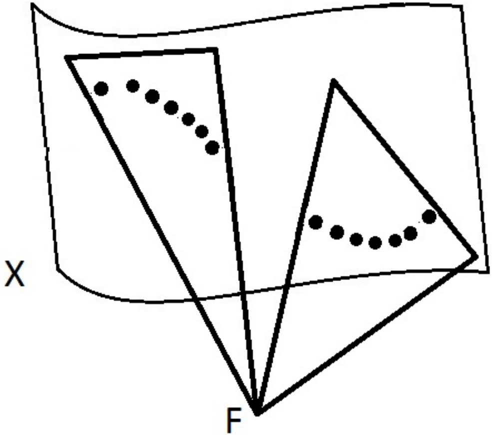

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The spans are two $\mathbb{P}^6$'s meeting at F.

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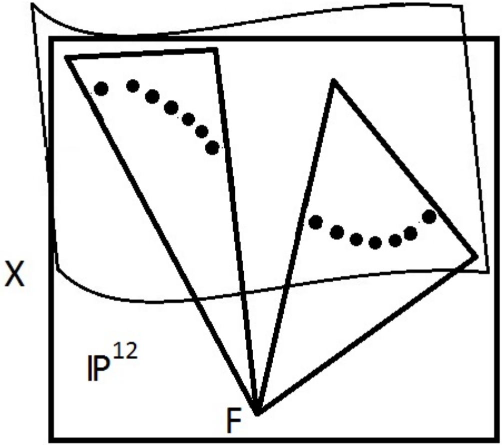

The spans are two $\mathbb{P}^6$'s meeting at F.
The two $\mathbb{P}^6$'s span a $\mathbb{P}^{12}$ and not a $\mathbb{P}^{13}$.

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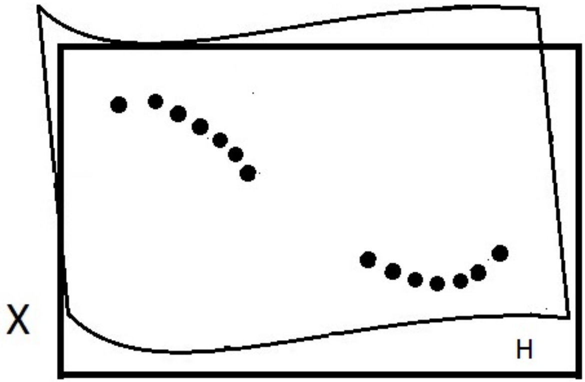

The 14 points span a $\mathbb{P}^{12}$ and not a $\mathbb{P}^{13}$.
They do not impose independent conditions to hyperplanes H.

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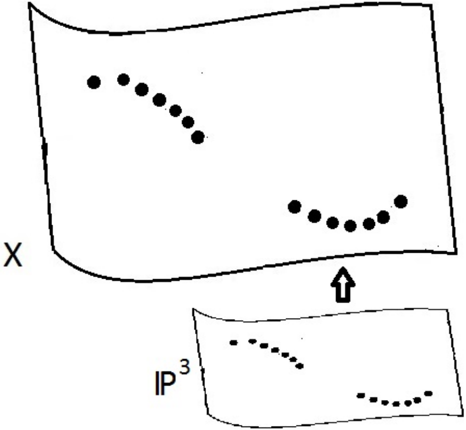

The pre-images of the 14 points in $\mathbb{P}^3$ do not impose independent conditions to quartics.

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The question is translated in terms of simple points interpolation in a projective space of dimension 3.

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Interpolation for simple points

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where R is the polynomial ring $R = \mathbb{C}[x_0, \ldots, x_n]$ and I is the homogeneous ideal of W (the subscripts indicate the degree).

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Hilbert function of W

$$H_W(d) = \dim(R_d / I_d).$$

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In our situation, the two decompositions of F determine a set $W = Z \cup Z' \subset \mathbb{P}^3$, of (at most) 14 points and not less than 10 points, which imposes only 7 conditions to **quadrics**. I.e. $H_W(2) \leq 7$.

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Castelnuovo's Theorem

If a set W of at least $n+6$ points in $\mathbb{P}^n$ imposes non more than $n+4$ conditions to quadrics, then the points are contained in a **rational normal curve** (a curve of degree n in $\mathbb{P}^n$).

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detect if Z sits in a rational normal curve (e.g. with algoritmhs of computer algebra).

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Since the dimension of the Terracini's tangent space can easily be detecyted by linear algebra, we get a rather effective method for deciding the uniquess and the rank of F, as soon as we heve a decomposition of F with 7 summands. (LC - Ottaviani - Vannieuwenhoven 2017).

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General non-sense, Ballico-LC 2012

When the rank is small ($k < 3d/2$) then F is identifiable, unless F has infinitely many decompositions.

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Similar problems for extensions to quintics, etc. (WORK IN PROGRESS).

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Work in progress with J. Migliore.

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- Comon's problem has a positive answer for *general* forms of degree 8 in 3 variables.

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Fix a partition $E \sqcup F = \{1, \ldots, k\}$ and call $Z(E)$ (resp. $Z(F)$) the projections of Z to the factor of the partition in E (resp. in F).
If $Z(E)$, $Z(F)$ are linearly independent in the corresponding Segre embedding, then the decomposition **is minimal**.

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WORK IN PROGRESS

Final remark

Thank you for your attention