On the tensor rank of networks of entangled pairs

tensor surgery and the laser method

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arXiv:1606.04085 and arXiv:1609.07476

Prelude

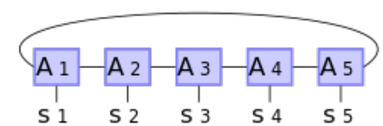
- Quantum Information
  Tensors = multiparticle quantum states
  Tensor rank = multiparticle entanglement

$$|000\rangle + |111\rangle$$

- Computer Science
  Tensors = description of algebraic problems
  Tensor rank = complexity

matrix multiplication
communication complexity

- Physics
  Tensor networks = description of ground states
  Tensor structure = physical properties

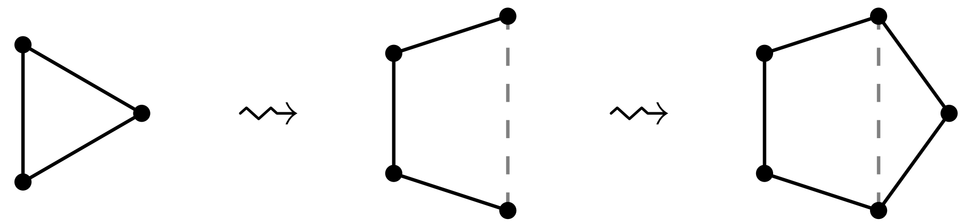


Wiki

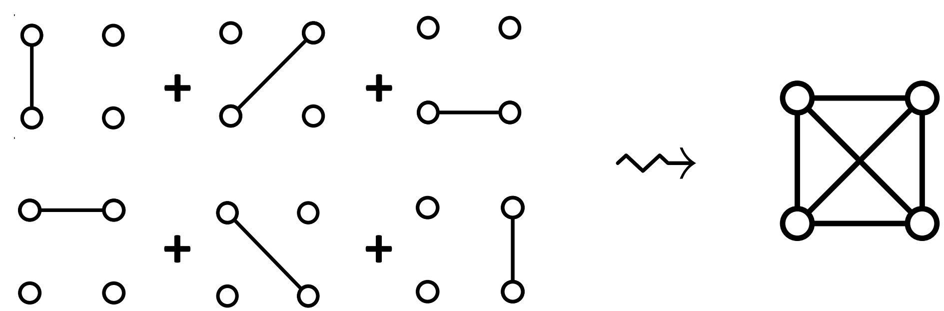
- Relativity Theory, Engineering, ...

Overview

- Tensors, rank and networks of entangled pairs
- Tensor Surgery



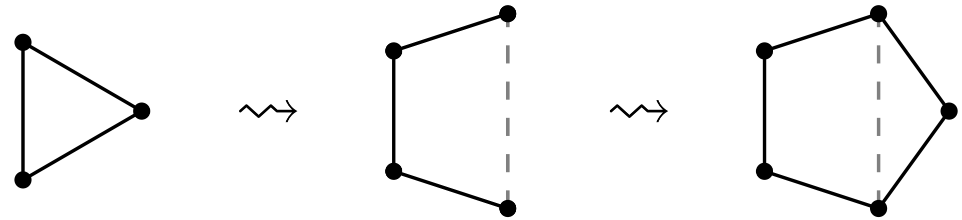
- Laser Method



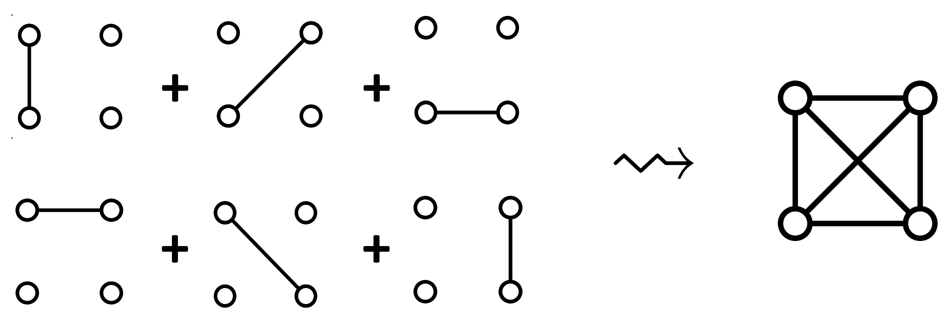
Overview

- <u>Tensors, rank and networks of entangled pairs</u>

- Tensor Surgery



- Laser Method



Tensors

tensor=quantum state
overall normalisation
does not matter

- k-tensor t is an element in $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$

$$b_{i_1} \otimes \cdots \otimes b_{i_n}$$

basis

$$b_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- k=1: vector $|+\rangle = |0\rangle + |1\rangle = b_0 + b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

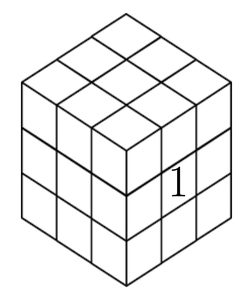
- k=2: matrix $|00\rangle + |11\rangle = b_0 \otimes b_0 + b_1 \otimes b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$b_{i_1} \otimes b_{i_2} = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & 1 \\ 0 & \ldots & 0 \end{pmatrix}$$

- k=3: cube $|001\rangle + |010\rangle + |100\rangle = b_0 \otimes b_0 \otimes b_1 + b_0 \otimes b_1 \otimes b_0 + b_1 \otimes b_0 \otimes b_0$

$$\begin{matrix} 0_1 & 1_0 \\ 1_0 & 0_0 \end{matrix}$$

$$b_{i_1} \otimes b_{i_2} \otimes b_{i_3} =$$



Resource theory of tensors

(Strassen 1991, Dür, Vidal & Cirac 2000)

- Given k-tensors s and t.
- Can s be transformed into t by local operations? Do matrices $A_1, \ldots, A_k$ exist, so that

SLOCC

$$A_1 \otimes \cdots A_k s = t \quad ?$$

- If yes, we write $s \geq t$.
- unit tensor (GHZ entangled state)

$$T_d(k) = \sum_{i=1}^{d} b_i \otimes \cdots \otimes b_i$$

Resource theory of tensors

- "entanglement cost" of a k-tensor t is the smallest d, s.th.

$$T_d(k) \geq t$$

- equals tensor rank R(t)

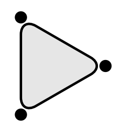
$$R(t) = \min\{d : t = \sum_i v_1^{(i)} \otimes \cdots \otimes v_k^{(i)}\}$$

- "distillable entanglement" of a k-tensor t is the largest d, s.th.

$$t \geq T_d(k)$$

equals subrank

Tensor rank

- k=2: tensor rank=matrix rank
  → easy to compute
  Example:
  Identity matrix (EPR state) $\mathrm{T}(\circ\!\!-\!\!-\!\!\circ) = \sum_{i\in\{0,1\}} b_i \otimes b_i \in \mathbb{C}^2 \otimes \mathbb{C}^2$
- k>2: NP hard (Håstad)
  Examples:
  Rank 2: GHZ state $\mathrm{T}\Big($  $\Big) = \sum_{i\in\{0,1\}} b_i \otimes b_i \otimes b_i \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

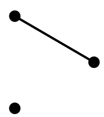
  Rank 3: W-state $b_0 \otimes b_0 \otimes b_1 + b_0 \otimes b_1 \otimes b_0 + b_1 \otimes b_0 \otimes b_0$

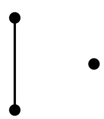
Tensor rank versus tensor product

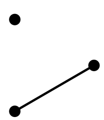
- Given s and t both k-tensors

$$R(s \otimes t) \leq R(s)R(t)$$

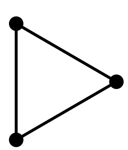
- Equality for 2-tensors, strict in general

$\mathrm{T}\Big($  $\Big) = \sum_{i\in\{1,2\}} b_i \otimes b_i \otimes 1 \; \in \; \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C},$

$\mathrm{T}\Big($  $\Big) = \sum_{i\in\{1,2\}} b_i \otimes 1 \otimes b_i \; \in \; \mathbb{C}^2 \otimes \mathbb{C} \otimes \mathbb{C}^2,$

$\mathrm{T}\Big($  $\Big) = \sum_{i\in\{1,2\}} 1 \otimes b_i \otimes b_i \; \in \; \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.$

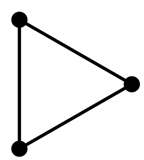
2x2 matrix multiplication tensor, rank =7 (Strassen)

$\mathrm{T}\Big($  $\Big) = \sum_{i,j,k\in\{1,2\}} (b_i \otimes b_j) \otimes (b_j \otimes b_k) \otimes (b_k \otimes b_i)$

Strassen's 7er

- Define $b_+ := b_0 + b_1$ and $b_- := b_0 - b_1$ in $\mathbb{C}^2$

  $b_{xy} := b_x \otimes b_y \in \mathbb{C}^2 \otimes \mathbb{C}^2$

T() =

$$
\begin{aligned}
&- b_{-0} \otimes b_{0+} \otimes b_{11} \quad - b_{11} \otimes b_{-0} \otimes b_{0+} \quad - b_{0+} \otimes b_{11} \otimes b_{-0} \\
&+ b_{-1} \otimes b_{1+} \otimes b_{00} \quad + b_{00} \otimes b_{-1} \otimes b_{1+} \quad + b_{1+} \otimes b_{00} \otimes b_{-1} \\
&+ (b_{00} + b_{11}) \otimes (b_{00} + b_{11}) \otimes (b_{00} + b_{11}).
\end{aligned}
$$

takes half an hour to verify

Border rank

- Sometimes, tensor of rank r can be approximated arbitrarily by tensors of rank b<r

$$(b_0 + \epsilon b_1)^{\otimes 3} - b_0^{\otimes 3}$$
$$= \epsilon(b_0 \otimes b_0 \otimes b_1 + b_0 \otimes b_1 \otimes b_0 + b_1 \otimes b_0 \otimes b_0) + O(\epsilon^2)$$

- Smallest b is called border rank $\underline{R}(t)$
- More generally, approximate transformation from s to t.

Resource theory of tensors

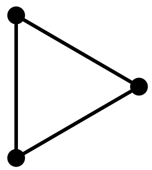
- Given k-tensors s and t.
  When is $s^{\otimes m} \geq t^{\otimes n}$?
- Best ratio m/n denoted by $\omega(s,t)$
- Asymptotic log rank $\omega(t) := \omega(T_2(k), t)$
- Asymptotic log subrank $q(t) := \omega(t, T_2(k))^{-1}$
- Theorem (Strassen & co):

$$q(t) \leq \omega(t) \leq \log \underline{R}(t) \leq \log R(t)$$

$$q(t) \leq \omega(t) \leq \log \underline{R}(t) \leq \log R(t)$$

matrix multiplication exponent

- Mamu

T()

$$q(t) = 2 \qquad \omega(t) \leq 2.38$$

Strassen

Coppersmith-Winograd, ... Le Gall

$\underline{R}(t) = 7$ $\quad R(t) = 7$

Landsberg

- W state

$$b_0 \otimes b_0 \otimes b_1 + b_0 \otimes b_1 \otimes b_0 + b_1 \otimes b_0 \otimes b_0$$

$$q(W) = h\left(\frac{1}{3}\right) \approx 0.92 \qquad \omega(t) = 1$$

Coppersmith-Winograd

$\underline{R}(t) = 2$ $\quad R(t) = 3$

Motivation for our work

- Log rank is a lower bound on the quantum communication complexity of a function f(x,y)
- Exact for non-deterministic case
  Equality game = unit tensor
  Pairwise equality (among 3) = Mamu
  Savings over the classical case: $\log_2 7 < 3$
- What about more players?
  Pairwise equality in a circle or graph.

with Buhrman and Zuiddam, 1603.03757
Proceedings of ITCS 2017

Benchmark: lower bound

- Rank, Border rank and asymptotic rank are decreasing under grouping of particles
- Group k particles into set S and complement

$$R(t) \geq R_S(t)$$

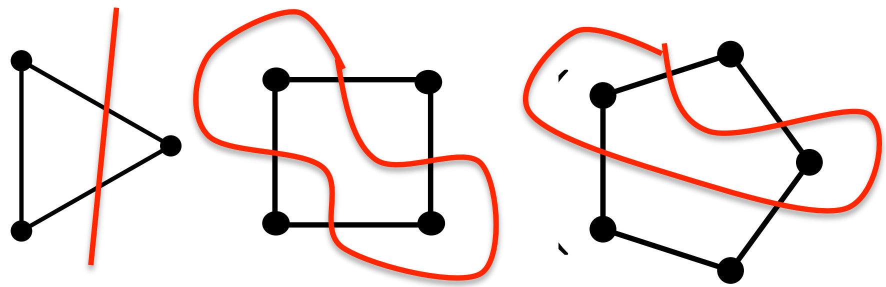
- this is matrix rank → easy to compute
- For a graph of entangled pairs

$$R_S(t) = 2^{\#\text{edges leaving} S} \qquad \omega(T(G)) \geq \text{maxcut} G$$

- Upper bound = # edges in graph

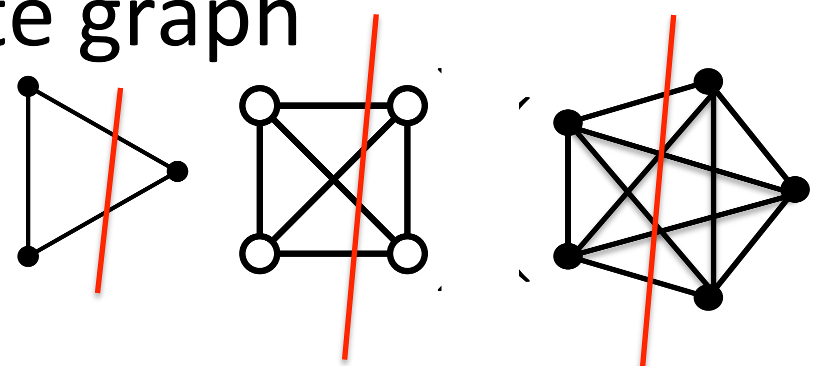
$$\#\text{edges} \geq \omega(T(G)) \geq \text{maxcut}\, G$$

Cycle graph



			k even	k odd
3	4	5	k	k
2	4	4	k	k-1

Complete graph

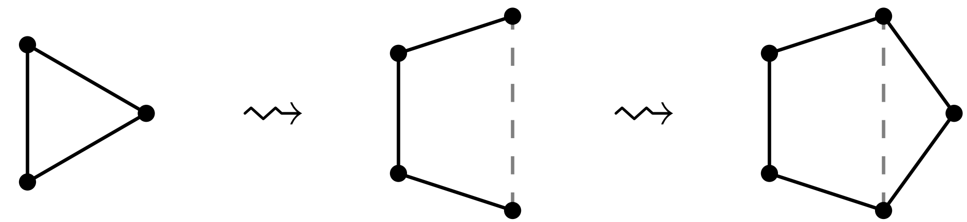


3	6	10	$\frac{k(k+1)}{2}$	$\frac{k(k+1)}{2}$
2	4	6	$\frac{k^2}{4}$	$\frac{k^2-1}{4}$

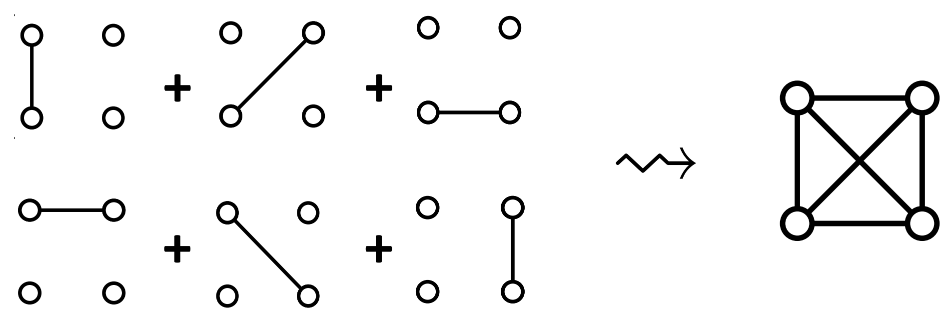
Can we match the lower bounds?

Overview

- Tensors, rank and networks of entangled pairs
- Tensor Surgery

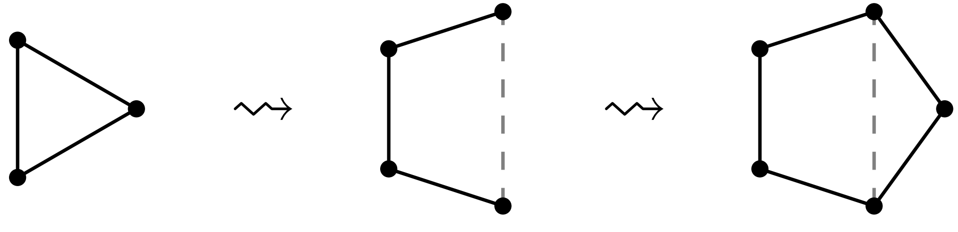


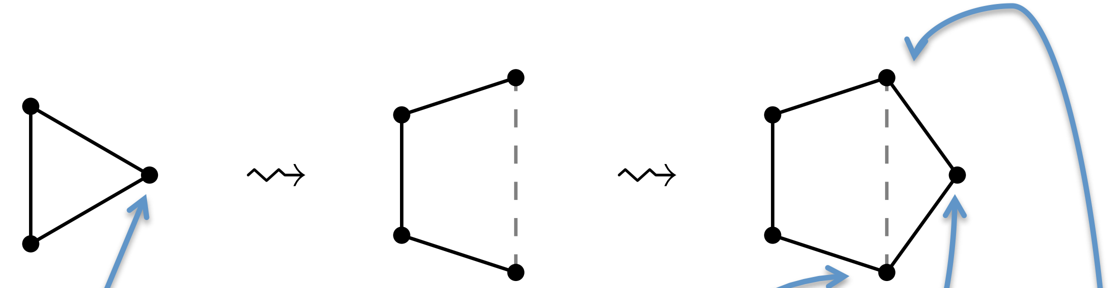
- Laser Method



Tensor surgery

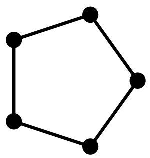
- Transform non-trivial decomposition of initial tensor into non-trivial decomposition of target tensor
- Example:
  rank 7 decomposition of 3 cycle
  → rank 31 decomposition of 5 cycle
- Procedure:
  1) cut open, generates virtual pair
  2) insert 2 pairs and absorb virtual pair





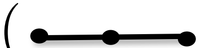
$$\phi : \mathbb{C}^2 \otimes \mathbb{C}^2 \to (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes 3}$$

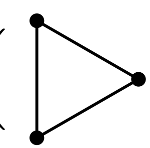
$$u \otimes v \mapsto \sum_{j \in \{0,1\}^2} (u \otimes b_{j_1}) \otimes (b_{j_1} \otimes b_{j_2}) \otimes (b_{j_2} \otimes v).$$

T()

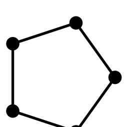
$$= -\phi(b_{-0}) \otimes b_{0+} \otimes b_{11} - \phi(b_{11}) \otimes b_{-0} \otimes b_{0+} - \phi(b_{0+}) \otimes b_{11} \otimes b_{-0}$$
$$+ \phi(b_{-1}) \otimes b_{1+} \otimes b_{00} + \phi(b_{00}) \otimes b_{-1} \otimes b_{1+} + \phi(b_{1+}) \otimes b_{00} \otimes b_{-1}$$
$$+ \phi(b_{00} + b_{11}) \otimes (b_{00} + b_{11}) \otimes (b_{00} + b_{11}).$$

Rank=4

$$\phi(b_{xy}) = \sum_{j \in \{0,1\}^2} (b_x \otimes b_{j_1}) \otimes (b_{j_1} \otimes b_{j_2}) \otimes (b_{j_2} \otimes b_y) =$$ $T($  $)$

$$\phi(b_{00} + b_{11}) = \sum_{i \in \{0,1\}^3} (b_{i_1} \otimes b_{i_2}) \otimes (b_{i_2} \otimes b_{i_3}) \otimes (b_{i_3} \otimes b_{i_1}) =$$ T()

Wow! Rank=7

$R($T( $)) \leq 6 \times 4 + 1 \times 7 = 31 < 2^5$

31<32 ...

- works for all k-cycles $\mathrm{R}(\mathrm{T}(C_k)) \leq 2^k - 1$
- works for border rank as well
- and asymptotic rank
  without knowledge of the decomposition,
  always using the "7" mamu upper bound

$$\omega_k \leq \omega_{k-2} + \omega_3$$

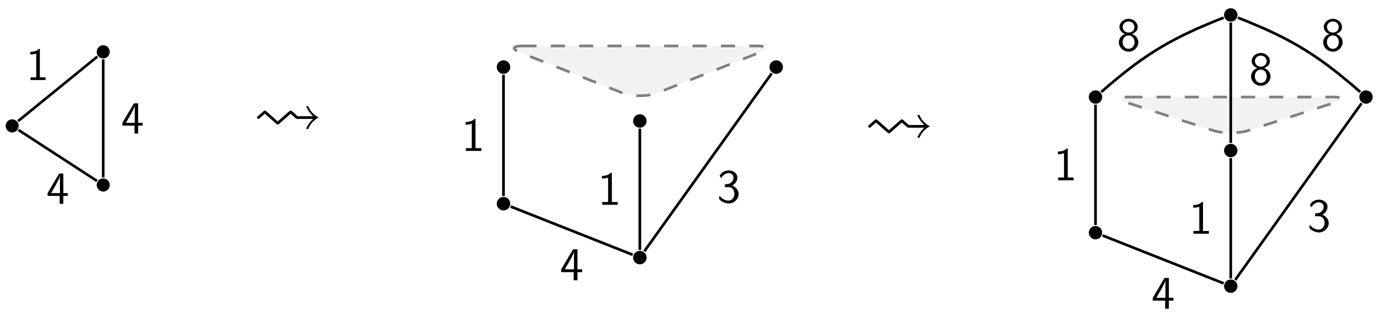
$$If\ \omega = 2,\ then\ \omega_k = k-1\ for\ all\ odd\ k$$

- uniformly bounded away from k

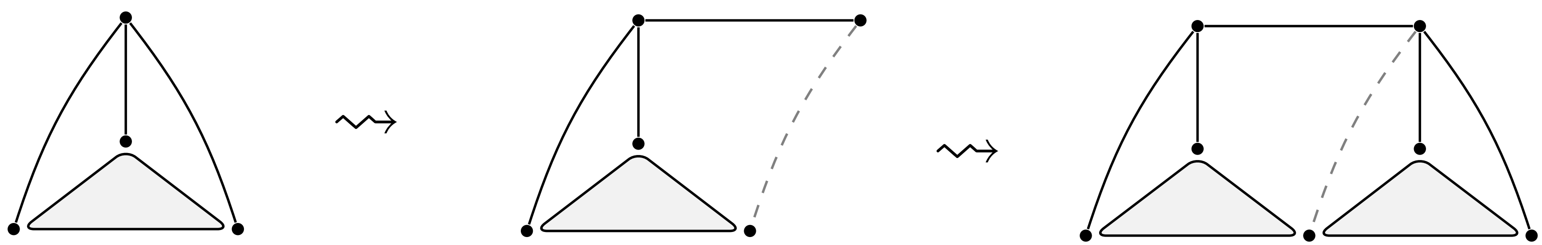
$$\omega_k \leq k - \alpha\left(1 + \frac{1-\alpha}{k-1+\alpha}\right) \leq k - \alpha$$

bigger than 0.3
(Le Gall)
=1 iff w=2

Other graphs and hypergraphs



$$\omega(\mathrm{T}(G)) = 32$$

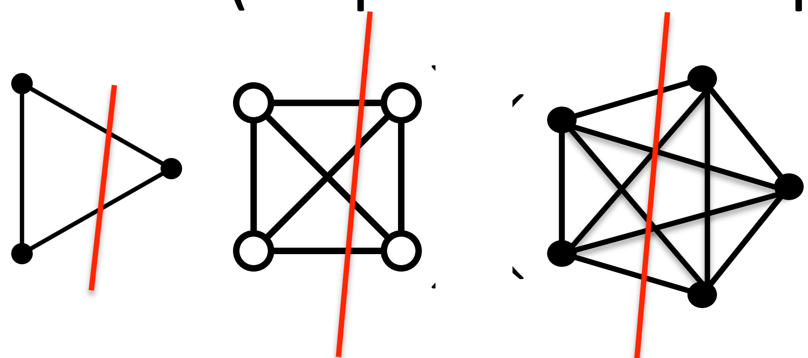


$$6 \leq \omega(\mathrm{T}(H)) \leq 6\omega/2.$$

Works well for sparse graphs and hypergraphs!
Your tensor?
What about dense graphs?

Dense graphs

- Tensor surgery needs a good starting tensor
  The best we have is mamu!
- in tensor sugergy asymptotic log rank per edge increases (no problem for sparse graphs)



2
2/3

4
4/6=2/3

6
6/10=3/5

$$\frac{k^2}{4}$$

$$\frac{k^2}{4} / \binom{k}{2} \approx \frac{1}{2}$$

- best upper bounds from mamu covering $2.38/3 \approx 0.79$
- Can we beat this for some graph?

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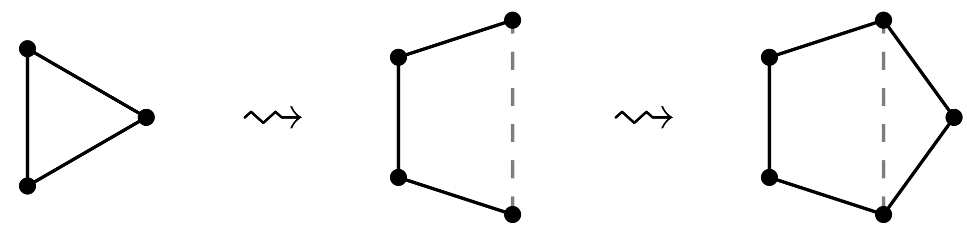
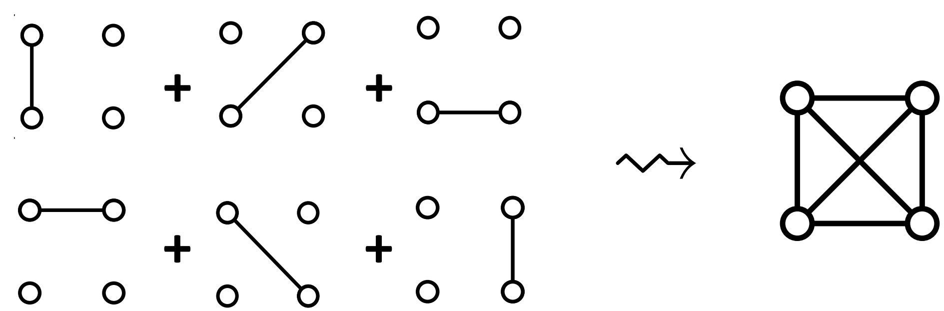


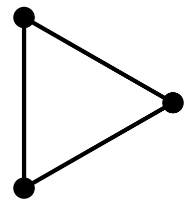
- Laser Method



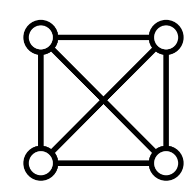
Laser method

- by Strassen for upper bound on mamu exponent
- with Coppersmith and Winograd starting tensor

1) choose starting tensor with low border rank
   with suitable coarse outer structure (W type)
   and fine inner structure (Mamu-type)
2) take many copies and extract unit tensors
   from W, each inner tensor is Mamu-type
3) The Mamu-type tensors can be a little different.
   Schönhage´s asymptotic sum inequality makes
   them equal (coherent – thus the name “laser”)

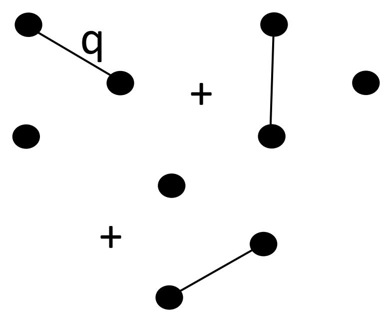


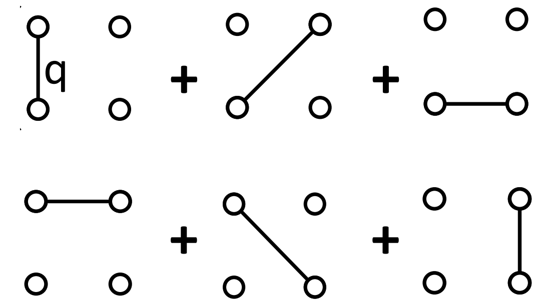
versus



starting tensor

$\underline{R} \leq q+2$



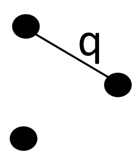


outer structure

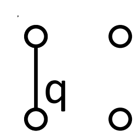
$W = b_1 \otimes b_1 \otimes b_2$
$+ \text{ permutations}$

$D_{2,2} = b_1 \otimes b_1 \otimes b_2 \otimes b_2$
$+ \text{ permutations}$

inner structure
(1 copy)



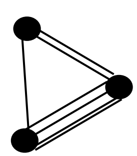
or ...



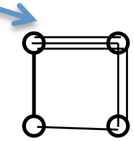
or ...

(many copies)

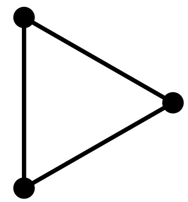
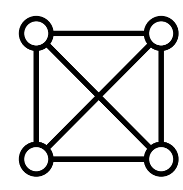
unbalanced



or ...



or ...

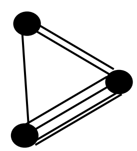
 versus 

extract diagonal

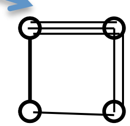
$W^{\otimes n} \geq (b_1 \otimes b_1 \otimes b_1 + b_2 \otimes b_2 \otimes b_2)^{\otimes nq(W)}$

$D_{2,2}^{\otimes n} \geq (b_1 \otimes b_1 \otimes b_1 \otimes b_1 + b_2 \otimes b_2 \otimes b_2 \otimes b_2)^{\otimes nq(D_{2,2})}$

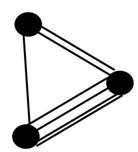
inner tensor

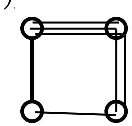
 or ...

unbalanced (n pairs)

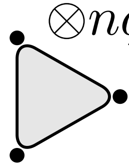
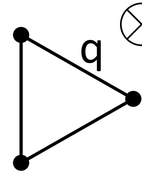
 or ...

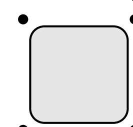
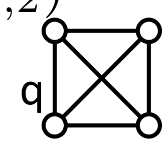
total tensor

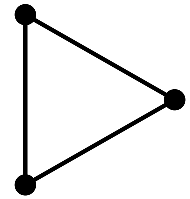
$\bigoplus_{i=1}^{2^{nq(W)}}$ 

$\bigoplus_{i=1}^{2^{nq(D_{2,2})}}$ 

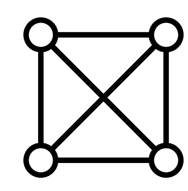
lasering

$\left(\right.$$\left.\right)^{\otimes nq(W)} \otimes \left(\right.$$\left.\right)^{\otimes (n/3)}$

$\left(\right.$$\left.\right)^{\otimes nq(D_{2,2})} \otimes \left(\right.$$\left.\right)^{\otimes (n/6)}$

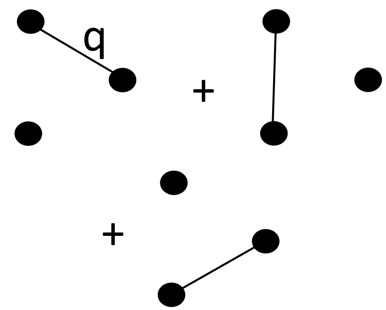


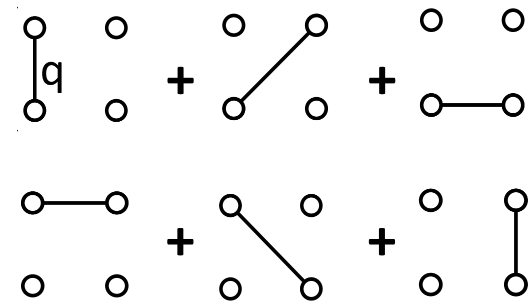
versus



starting tensor

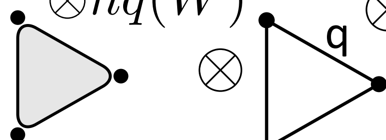
$\underline{R} \leq q + 2$



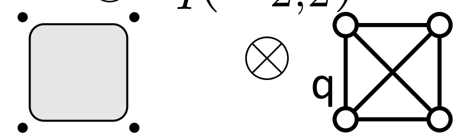


final tensor

$\otimes nq(W)$ $\otimes$ $\otimes(n/3)$



$\otimes nq(D_{2,2})$ $\otimes$ $\otimes(n/6)$



$R \leq (q+2)^n$

Coppersmith-Winograd = $h\left(\frac{1}{3}\right) \approx 0.92$

we show=1

goal tensor

$$\tau \leq \log_q(q+2) - \log_q(2)q(W)$$

$$\tau \leq \log_q(q+2) - \log_q(2)q(D_{2,2})$$

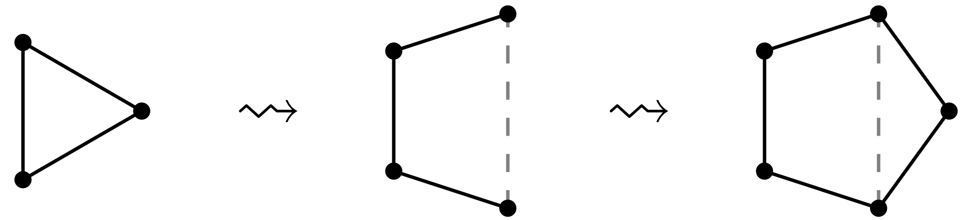
best known

$\tau \leq 0.79$ ← Le Gall

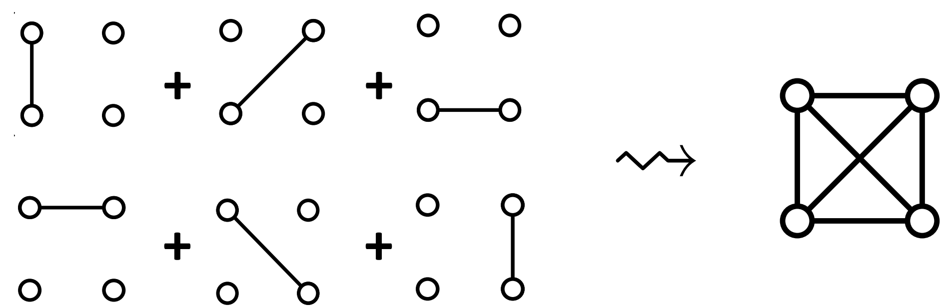
$\tau \leq 0.77$ ← optimise over q

Summary

- Tensor Surgery



nontrivial rank results for k-cycle
optimal asymptotic rank results
good for sparse graphs



- Laser Method

beating matrix multiplication for best asymptotic rank per edge

Advertisement

- Tensor rank is not multiplicative under the tensor product
- with Asger Kjærulf-Jensen & Jeroen Zuiddam
- arxiv:1705.09379
- Main Results:

3-tensor

$$R(\underbrace{W \otimes W}_{6-tensor}) \leq 8 < 9 = R(W)^2$$

$$R(t^{\otimes n}) \leq \text{poly}(n)\underline{R}(t)^n$$