Entanglement, auxiliary varieties and simple singularities

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Entanglement of pure quantum states

- $\mathcal{H} = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$, Hilbert space for a n-partite system
- $|\psi\rangle \in \mathbf{P}(\mathcal{H})$, a quantum pure state
- $G = \text{SLOCC} = SL_{d_1}(\mathbb{C}) \times \cdots \times SL_{d_n}(\mathbb{C})$, reversible local operations
- $X_{Sep} = \{|\psi\rangle \in \mathbf{P}(\mathcal{H}), |\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle\}$, the set of separable states
- $\mathbf{P}(\mathcal{H}) \setminus X_{Sep}$, the set of entangled states

The set X_{Sep} is well-known to geometers as the Segre variety

$$\begin{array}{cccc} Seg: & \mathbf{P}^{d_1-1} \times \cdots \times \mathbf{P}^{d_n-1} & \mapsto & \mathbf{P}(\mathcal{H}) \\ & (v_1, \ldots, v_n) & \mapsto & [v_1 \otimes \cdots \otimes v_n] \end{array} \quad (1)$$

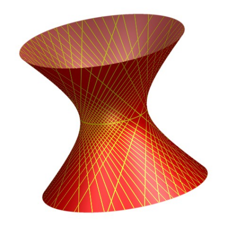
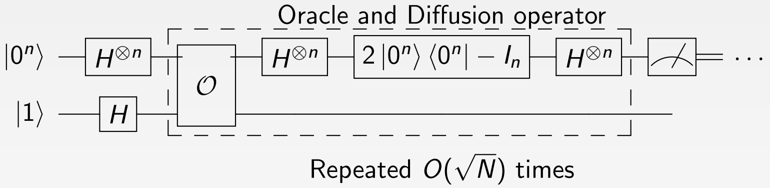


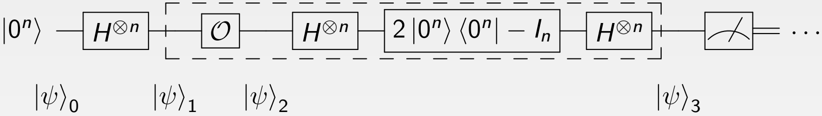
Figure : $Seg(\mathbf{P}^1 \times \mathbf{P}^1) \subset \mathbf{P}^3$

Grover's quantum algorithm

Problem: Find a "marked" element in an unsorted database of $N = 2^n$ items. Grover'algorithm does it in $O(\sqrt{N})$ (instead of $O(N)$ classicaly).



- The gate $\mathcal{O}$ (for Oracle) signs the marked element $\mathcal{O}(|x\rangle) = |x\rangle$ and $\mathcal{O}(|x_0\rangle) = -|x_0\rangle$
- The Diffusion operator symmetrizes the amplitudes of the state with respect to the mean value of the amplitudes



Evolution of Grover's algorithm

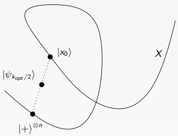
- $|\psi\rangle_0 = |0\dots 0\rangle$
- $|\psi\rangle_1 = \frac{1}{\sqrt{N}}\sum_{(i_1,\dots,i_N)\in\{0,1\}^N} |i_1\dots i_N\rangle$
- $|\psi\rangle_2 = -\frac{1}{\sqrt{N}}|j_1\dots j_N\rangle + \frac{1}{\sqrt{N}}\sum_{(i_1,\dots,i_N)\in\{0,1\}^N\setminus(j_1,\dots,j_N)} |i_1\dots i_N\rangle$
- $|\psi\rangle_3 = \alpha|j_1\dots j_N\rangle + \beta\sum_{(i_1,\dots,i_N)\in\{0,1\}^N\setminus(j_1,\dots,j_N)} |i_1\dots i_N\rangle$ with $|\alpha|^2 > |\beta|^2$
- After $\approx \frac{\sqrt{N}}{2}$ rounds we have $|\alpha|^2 >> |\beta|^2$.

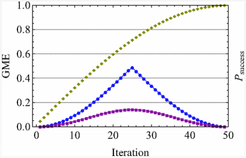
Grover's quantum algorithm

Where is the geometry ?

$$\begin{aligned}
|\psi\rangle_k &= \alpha_k |j_1 \dots j_N\rangle + \beta_k \textstyle\sum_{(i_1,\dots,i_N)\in\{0,1\}^N\setminus(j_1,\dots,j_N)} |i_1 \dots i_N\rangle \\
&= (\alpha_k - \beta_k) |j_1 \dots j_N\rangle + \beta_k \textstyle\sum_{(i_1,\dots,i_N)\in\{0,1\}^N} |i_1 \dots i_N\rangle \\
&= \tilde{\alpha}_k |j_1 \dots j_N\rangle + \tilde{\beta}_k |+\rangle^{\otimes n}
\end{aligned}$$

For one marked element, the states generated by Grover's algorithm $|\psi\rangle_k$ are rank two tensors. As pointed out by J.-L. Brylinski[1], the rank can be considered as an algebraic measure of entanglement. It gives qualitative interpretation of Rossi et al.[2]



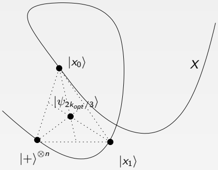


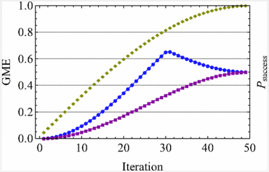
[1]Brylinsky, J.-L. *Algebraic measures of entanglement* in Mathematics of Quantum Computation. Comput. Math. Ser. Chapman and Hall (2002)

[2]Rossi, M., D. Bru, and C. Macchiavello. *Scale invariance of entanglement dynamics in Grover's quantum search algorithm*. Physical Review A 87.2 (2013): 022331.

Grover's algorithm

We can similarly address the question of the rank for multiple searched items
For example if the two searched elements have maximum Hamming distance then the rank is equal to the number of marked elements + 1





For one marked element running Grover's algorithm means moving on a secant line between $|+\rangle^{\otimes n}$ to the marked element, for two marked elements this corresponds to moving on a secant plane passing through $|+\rangle^{\otimes n}$ and the searched elements

Secant and Tangent varieties

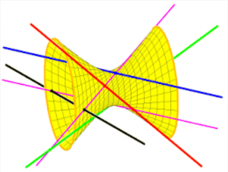
Given $X \subset \mathbf{P}(V)$ one can construct auxiliary varieties from X

- The Secant variety:

$$\sigma(X) = \overline{\cup_{x,y\in X}\mathbf{P}^1_{xy}} \quad (2)$$

- The Tangent variety:

$$\tau(X) = \cup_{x\in X} T_x X \quad (3)$$



- Higher secant can be also defined $\sigma_k(X) = \overline{\cup_{x_1,\dots,x_k\in X}\mathbf{P}^{k-1}_{x_1,\dots,x_k}}$
- More generally on has the notion of join of algebraic varieties, if $Y \subset X$ then $J(Y,X) = \overline{\cup_{x\in X, y\in Y}\mathbf{P}^1_{xy}}$

Tangent and Secant varieties in QIT

- $X_{Sep} = \mathbf{P}^{d_1-1} \times \cdots \times \mathbf{P}^{d_n-1}$ is SLOCC-homogeneous so by construction the auxiliary varieties are SLOCC invariants. Secant varieties (join, tangent) can be used to stratify the ambiant Hilbert space by SLOCC invariant varieties[3]
- $\sigma(X_{Sep}) = \overline{SLOCC.|GHZ\rangle}$ and $\tau(X_{Sep}) = \overline{SLOCC.|W\rangle}$ where $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\dots0\rangle + |1\dots1\rangle)$ and $|W\rangle = \frac{1}{\sqrt{n}}(|10\dots0\rangle + |01\dots0\rangle + \cdots + |0\dots01\rangle)$
- $\tau(X) \subset \sigma(X)$ and if $\dim(\sigma(X)) = 2\dim(X)+1$ then $\dim(\tau(X)) = 2\dim(X)$ (otherwise $\tau(X) = \sigma(X)$). This classical result is a geometrical way of saying that "three qubits can be entangled in two inequivalent ways[4]"

$$\underbrace{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1}_{X_{Sep}} \subset \cup_{1\leq i<j\leq 3} \underbrace{\mathbf{P}^3_{ij} \times \mathbf{P}^1}_{\text{Bisperable}} \subset \underbrace{\tau(X_{Sep})}_{|W\rangle} \subset \underbrace{\sigma(X_{Sep}) = \mathbf{P}^7}_{|GHZ\rangle} \quad (4)$$

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[3] Heydari, H. *Geometrical of entangled states and the secant variety*. Quantum Inf. Processing 7(1) (2008)
H- F., Luque J.-G., Thibon J.-Y. *Geometric descriptions of entangled states by auxiliary varieties*. Journal of Mathematical Physics 53.10 (2012): 102203.

[4] Dür, W., Vidal, G., and Cirac, J. I. (2000). *Three qubits can be entangled in two inequivalent ways*. Physical Review A, 62(6), 062314.

Three-partite entanglement

The orbit structure of the three qubit classification appears in others physical systems

$\mathcal{H}$	$SLOCC$	QIT interpretaion	$X_{Sep} \subset \mathbf{P}(\mathcal{H})$	$\mathfrak{g}$
$Sym^3(\mathbb{C}^2)$	$SL_2(\mathbb{C})$	Three bosonic qubit (2007)	$v_3(\mathbf{P}^1) \subset \mathbf{P}^3$	$\mathfrak{g}_2$
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$	$SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$	Three qubit (2001)	$\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^7$	$\mathfrak{so}_8$
$\bigwedge^{\langle 3 \rangle} \mathbb{C}^6$	$Sp_6(\mathbb{C})$	Three fermions with with 6 single particules state with a symplectic condition	$LG(3,6) \subset \mathbf{P}^{13}$	f_4
$\bigwedge^3 \mathbb{C}^6$	$SL_6(\mathbb{C})$	Three fermions with with 6 single particles state (2008)	$G(3,6) \subset \mathbf{P}^{19}$	$\mathfrak{e}_6$
Δ_{12}	$Spin(12)$	Particles in Fermionic Fock space (2014)	$\mathbb{S}_6 \subset \mathbf{P}^{31}$	$\mathfrak{e}_7$
V_{56}	E_7	Three partite entanglement of seven qubit (2007)	$E_7/P_1 \subset \mathbf{P}^{56}$	$\mathfrak{e}_8$
			Freudenthal sub-adjoint varities[5]	

[5] Landsberg, J. M., Manivel, L., *The projective geometry of Freudenthal's magic square*. Journal of Algebra 239.2 (2001): 477-512.

Fermionic Fock space and the spinor representation

V a N dimensional complex vector space corresponding to one particule states

$$\mathcal{F} = \wedge^\bullet V = \mathbb{C} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^N V = \underbrace{\wedge^{even} V}_{\mathcal{F}_+} \oplus \underbrace{\wedge^{odd} V}_{\mathcal{F}_-} \quad (5)$$

The vector space is generated from the Vaccum $|0\rangle$ (a generator of $\wedge^0 V$) by applying creation operators $\mathbf{p}_i$, $1 \leq i \leq N$, i.e. a state $|\psi\rangle \in \mathcal{F}$ is given by

$$|\psi\rangle = \sum_{i_1,\dots,i_k} \psi_{i_1,\dots,i_k} \mathbf{p}_{i_1} \dots \mathbf{p}_{i_k} |0\rangle \text{ with } \psi_{i_1,\dots,i_k} \text{ skew symmetric tensors} \quad (6)$$

The annihilation operators $\mathbf{n}_j$, $1 \leq j \leq N$ are defined such that $\mathbf{n}_j |0\rangle = 0$ and

$$\{\mathbf{p}_i, \mathbf{n}_j\} = \mathbf{p}_i \mathbf{n}_j + \mathbf{n}_j \mathbf{p}_i = \delta_{ij}, \{\mathbf{p}_i, \mathbf{p}_j\} = 0, \{\mathbf{n}_i, \mathbf{n}_j\} = 0 \quad (7)$$

Considering $W = V \oplus V'$ where V and V' are isotropic subspaces, with basis $(e_j)_{1\leq j\leq 2N}$, for the quadratic form $Q = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}$
$\mathcal{F}$ is a $Cl(W, Q)$ module with $w = x_i e_i + y_j e_{N+j} \mapsto \sqrt{2}(x_i \mathbf{p}_i + y_j \mathbf{n}_j) \in End(\mathcal{F})$

Embedding qubits in Fermionic Fock space

$\mathcal{F}_+$ and $\mathcal{F}_-$ are $Spin(2N)$ irreducible representations (spinor representations) $G = Spin(2N)$ can be considered as a generalization of the SLOCC group[6]. Indeed on can embedd bosonic, qubit and fermionic systems into $\mathcal{F}_\pm$ such that the subgroup of G stabilizing the embedded Hilbert space is the usual SLOCC group

Example (The box picture)

Let $V = \mathbb{C}^{2n} = \mathbb{C}^2 \otimes \mathbb{C}^n$ *and* $\mathbf{p}_1, \dots, \mathbf{p}_n, \mathbf{p}_{\bar{1}}, \dots, \mathbf{p}_{\bar{n}}$ *be the creation operators. One can give a box picture of embedding of n-qubit in the Hilbert space* $\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$ *where the indices* i *represent some location and* $\{i, \bar{i}\}$ *a two level system*

$\square$	$\square$	$\square$	$\cdots$	$\square$	$\vert 000\dots 0\rangle$		$\uparrow$	$\uparrow$	$\uparrow$	$\cdots$	$\uparrow$	$\vert 000\dots 0\rangle$
$\downarrow\uparrow$	$\square$	$\square$	$\cdots$	$\square$	$\vert 100\dots 0\rangle$		$\downarrow$	$\uparrow$	$\uparrow$	$\cdots$	$\uparrow$	$\vert 100\dots 0\rangle$
$\downarrow\uparrow$	$\downarrow\uparrow$	$\square$	$\cdots$	$\square$	$\vert 110\dots 0\rangle$		$\downarrow$	$\downarrow$	$\uparrow$	$\cdots$	$\uparrow$	$\vert 110\dots 0\rangle$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\downarrow\uparrow$	$\downarrow\uparrow$	$\downarrow\uparrow$	$\cdots$	$\downarrow\uparrow$	$\vert 111\dots 1\rangle$		$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$	$\vert 111\dots 1\rangle$

[6] Sárosi, G. Lévay, P., *Entanglement in fermionic Fock space*. Journal of Physics A: Mathematical and Theoretical 47.11 (2014): 115304. Lévay, P., H- F., *Embedding qubits into fermionic Fock space: Peculiarities of the four-qubit case* Physical Review D 91.12 (2015): 125029.

Dual variety

Another auxilary variety of interest is the dual variety of X

$$X^* = \overline{\{H \in (\mathbf{P}^N)^*, \exists x \in X, T_x X \subset H\}} \quad (8)$$

X^* parametrizes the set of singular hyperplanes

For $X = \mathbf{P}^{d_1-1} \times \mathbf{P}^{d_2-1} \times \cdots \times \mathbf{P}^{d_n-1}$ (with $d_j \leq \sum_{i \neq j} d_i$), the variety X^* is always a hypersurface and by construction this hypersurface is SLOCC-invariant. The singular locus of this hypersurface is also SLOCC invariant. It can be used to stratify the (projectivized) Hilbert space

This idea goes back to Miyake[7] who used in his paper previous results of Weyman and Zelevinksy on singularities of hyperdeterminant[8]. Following Miyake the three-qubit classification becomes:

$$\underbrace{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1}_{X_{Sep}} \subset \underbrace{X^*_{sing} = \cup_{1 \leq i \leq 3} X^*_{i,sing}}_{\text{Biseparable}} \subset \underbrace{X^*}_{|W\rangle} \subset \underbrace{\mathbf{P}^7}_{|GHZ\rangle} \quad (9)$$

[7] Miyake, A. *Classification of multipartite entangled states by multidimensional determinant* Phys. Rev A 67 012108 (2003)

[8] Weyman, J., Zlevinsky, A., *Singularities of hyperdeterminants*, Ann. Institut Fourier 46 (1993)

Singular hyperplane sections

This idea of Miyake can be investigated further by looking at what type of singular hyperplane section can be associated to a state. Let $\mathcal{H} = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ and $X_{Sep} = \mathbf{P}^{d_1-1} \times \cdots \times \mathbf{P}^{d_n-1}$. To a state $|\varphi\rangle \in \mathbf{P}(\mathcal{H})$ one associates the hypersurface of X_{Sep} given by

$$X_{Sep} \cap H_{\langle\varphi|} \subset X_{Sep} \tag{10}$$

where $H_{\langle\varphi|} = \{|\psi\rangle \in \mathbf{P}(\mathcal{H}), \langle\varphi, \psi\rangle = 0\}$
Using the rationnal map of the Segre embedding one can explicitely obtain the equation of the hyperplane section

$$\begin{array}{ccc} \mathbf{P}^{d_1-1} \times \cdots \times \mathbf{P}^{d_n-1} & \to & \mathbf{P}(\mathcal{H}) \\ ([x_1^1 : \cdots : x_{d_1}^1], \ldots, [x_1^n : \cdots : x_{d_n}^n]) & \mapsto & [x_1^1 x_1^2 \ldots x_1^n : \cdots : X^J : \cdots : x_{d_1}^1 x_{d_2}^2 \ldots x_{d_n}^n] \end{array} \tag{11}$$

Thus to a state $|\varphi\rangle = \sum a_{i_1 \ldots i_n} |i_1 \ldots i_n\rangle$ one associates the hypersurface of X_{Sep} defined by

$$f_{|\varphi\rangle} = \sum_{i_1, \ldots, i_n} a_{i_1 \ldots i_n} x_{i_1}^1 \ldots x_{i_n}^n \tag{12}$$

Singularity of hypersurfaces

If $|\varphi\rangle \in X^*$ then $f_{|\varphi\rangle}$ is a singular homogeneous polynomial, i.e. there exists $\overline{x}$ such that

$$f_{|\varphi\rangle}(\overline{x}) = 0 \text{ and } \partial_{i_k} f_{|\varphi\rangle}(\overline{x}) = 0 \quad (13)$$

In the 70s Arnol'd classified "simple singularities" of complex functions

One says that $(f_{|\varphi\rangle}, \overline{x})$ is simple iff under a small perturbation it can only degenerate to a finite number of non-equivalent singular hypersurfaces $(f_{|\varphi\rangle} + \varepsilon g, \overline{x}')$ (up to biholomorphic change of coordinates)

Simple singularities are always isolated, i.e. the Milnor number $\mu = \dim \mathbb{C}[x_1, \dots, x_n]/(\nabla f_{\overline{x}})$ is finite, and they can be classified in 5 families

Type	A_k	D_k	E_6	E_7	E_8
Normal form	$x^{k+1} + y^2$	$x^{k-1} + xy^2$	$x^3 + y^4$	$x^3 + xy^3$	$x^3 + y^5$
Milnor number	k	k	6	7	8

(14)

The singular type can be identified by computing the Milnor number, the corank of the Hessian and the cubic term in the degenerate directions

Example

Let us consider the 4-qubit state $|\psi\rangle = |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle$. The parametrization of the variety of separable state is given by
$\phi([x_0 : x_1], [y_0 : y_1], [z_0 : z_1], [t_0 : t_1]) = [x_0 y_0 z_0 t_0 : \cdots : x_1 y_1 z_1 t_1]$
The homogeneous polynomial associated to $|\psi\rangle$ is

$$f_{|\psi\rangle} = x_0 y_0 z_0 t_0 + x_1 y_0 z_1 t_1 + x_1 y_1 z_0 t_1 + x_1 y_1 z_1 t_0$$

In the chart $x_0 = y_1 = z_1 = t_1 = 1$ one obtains locally the hypersurface defined by

$$f(x, y, z, t) = yzt + xy + xz + xt$$

$(0, 0, 0, 0)$ is the only singular point of $f_{|\psi\rangle}$ (the hyperplane section is tangent to $|0111\rangle$).
The Hessian matrix of this singularity has co-rank 2 and $\mu = 4$.

$$(X \cap H_{|\psi\rangle}, |0111\rangle) \sim D_4$$

Entangled 4-qubit and the D_4 singularity

The 4-qubit system shows an infinite number of orbits. Those SLOCC orbits have been classified by Verstraete et al.[9]

Theorem (H-, Luque, Planat, JPhysA 2014)

Let H_φ be a hyperplane of $\mathbf{P}(\mathcal{H})$ tangent to $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^{15}$ and such that $X \cap H_\varphi$ has only isolated singular points. Then the singularities are either of types A_1, A_2, A_3, or of type D_4, and there exist hyperplanes realizing each type of singularity. Moreover, if we denote by $\widehat{X}^* \subset \mathcal{H}$ the cone over the dual variety of X, i.e. the zero locus of the Cayley hyperdeterminant of format $2 \times 2 \times 2 \times 2$, then the quotient map $\Phi : \mathcal{H} \to \mathbb{C}^4$ is such that $\Phi(\widehat{X}^*) = \Sigma_{D_4}$, where Σ_{D_4} is the discriminant of the miniversal deformation of the D_4-singularity.

Algorithm (H-, Luque, Thibon, JMP2017)

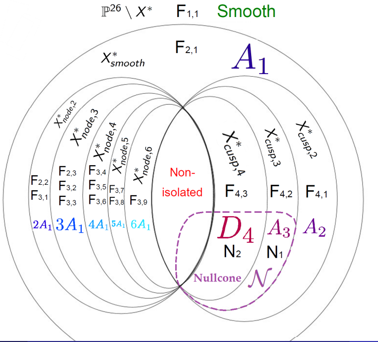
A covariant based algorithm to identify a four-qubit state with its normal form

[9] Verstraete, F., Dehaene, J., De Moor, B., and Verschelde, H. (2002). *Four qubits can be entangled in nine different ways*. Physical Review A, 65(5), 052112.

Entangled 3-qutrit and the D_4 singularity

Theorem (H-, Jaffali, Journal of Physics A 2016)

Let $H_\varphi \cap X$ be a singular hyperplane section of the algebraic variety of separable states for three-qutrit systems, i.e. $X = \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^{26}$ defined by a quantum pure state $|\varphi\rangle \in \mathbf{P}^{26}$. Then $H_\varphi \cap X$ only admits simple or nonisolated singularities. Moreover if x is an isolated singular point of $H_\varphi \cap X$, then its singular type is either A_1, A_2, A_3 or D_4.



Conclusion

- The language of algebraic geometry and representation theory provide a meaningful frame for studying SLOCC entanglement classes of pure states
- Fermionic Fock space (Spin representation) is a natural space to embedd all usual pure quantum systems (bosons, qubits, fermions). In this context the classification of entanglement classes corresponds to the classification of spinors
- Looking at entanglement classes from singularity theory prespective introduces new correspondence between Dynkin diagrams and SLOCC orbits classification

Perspectives

- Define quantum computation scheme within Fermionic Fock space
- Explore the meaning of Arnol'd adjency diagrams in terms of perturbation of states
- Study entanglement classes generated fby quantum algorithms by means of algebraic geometry: Can we prove that some states can not be generated by Grover's algorithm ?