Quantum computation with Turaev-Viro codes

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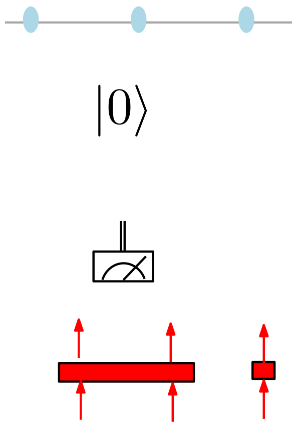
  - ground states
  - mapping class group representations
  - protected gates

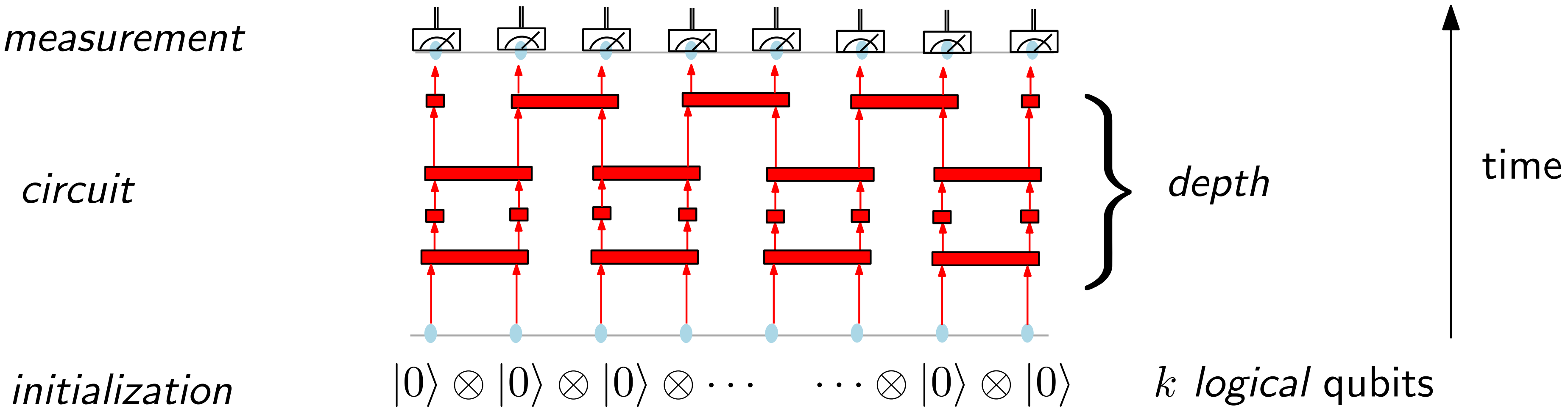
Quantum fault-tolerance: the DiVincenzo criteria

DiVicenzo criteria for fault-tolerant quantum computation

1. scalable physical system with well-characterized qubits
2. ability to initialize fiducial state
3. decoherence times $\gg$ gate operation time
4. qubit-specific measurement capability
5. universal set of quantum gates



David DiVincenzo



Quantum noise on n qubits

Quantum noise on n qubits is represented by a completely positive trace-preserving map (CPTPM)

$$\mathcal{N} : \mathcal{B}((\mathbb{C}^2)^{\otimes n}) \rightarrow \mathcal{B}((\mathbb{C}^2)^{\otimes n})$$

Operational problem: can we recover information subjected to such noise?

Using the Kraus decomposition $\mathcal{N}(\rho) = \sum_{E \in \mathcal{E}} E \rho E^\dagger$

it can be shown that it suffices to protect against against a certain set of errors $\mathcal{E}$

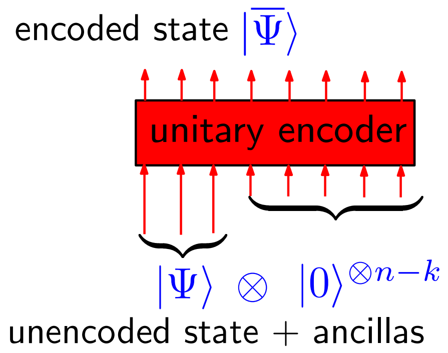
where an error is a linear map $E : (\mathbb{C}^2)^{\otimes n} \rightarrow (\mathbb{C}^2)^{\otimes n}$

Mathematical problem: Is there a recovery CPTPM $\mathcal{R} : \mathcal{B}((\mathbb{C}^2)^{\otimes n}) \rightarrow \mathcal{B}((\mathbb{C}^2)^{\otimes n})$

such that for ``suitable'' ρ $\quad \mathcal{R}(E \rho E^\dagger) \propto \rho \quad$ for all $E \in \mathcal{E}$

Procedure: (isometrically) embed/ "encode"

$$\begin{array}{ccc} (\mathbb{C}^2)^{\otimes k} & \rightarrow & \mathcal{L} \subset (\mathbb{C}^2)^{\otimes n} \\ \Psi & \mapsto & \overline{\Psi} \end{array}$$

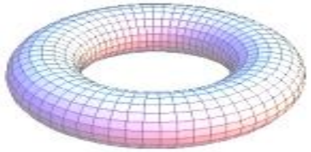


QEC condition:
[Knill, Laflamme]

$\mathcal{L}$ protects against errors $\mathcal{E}$ $\quad \Leftrightarrow \quad$ $\langle \overline{\Psi} | E^\dagger F | \overline{\varphi} \rangle = c(E,F) \langle \overline{\Psi} | \overline{\varphi} \rangle$

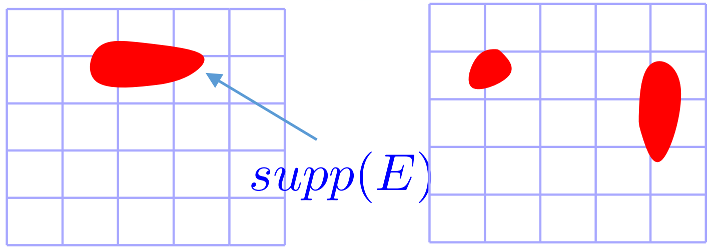
for all $E, F \in \mathcal{E}$, $\overline{\Psi}, \overline{\varphi} \in \mathcal{L}$

``Topological’’ error-correcting codes

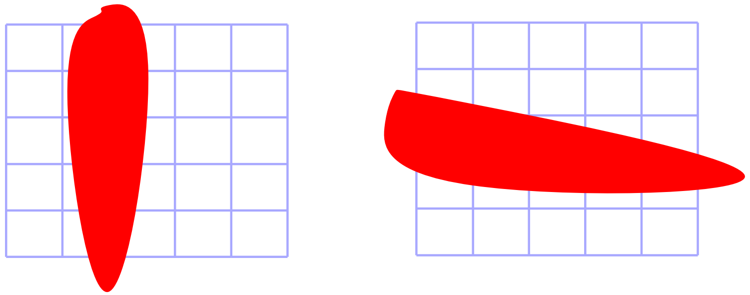


Def: A **“topological” code**:

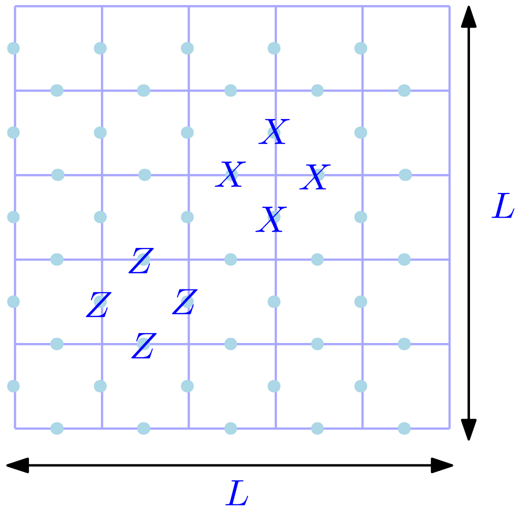
protects against all local errors, e.g.,
and more generally errors with
“topologically trivial” support



does not protect against errors
with topologically non-trivial
support, e.g.,



Example: Kitaev’s toric code



$n = 2L^2$ qubits on the edges of a edges of a $L \times L$ periodic lattices

$$\mathcal{L} = \{\Psi \in (\mathbb{C}^2)^{\otimes n} \,|A_v\Psi = B_p\Psi = \Psi \quad \text{for all } v,p\}$$

$A_v = X^{\otimes 4}$ for each vertex v

$B_p = Z^{\otimes 4}$ for each plaquette p

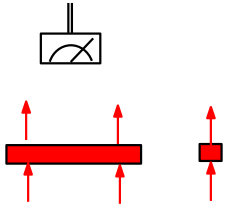
$k = \log_2 \dim \mathcal{L} = 2$ encoded qubits

Quantum fault-tolerance: the DiVincenzo criteria

DiVicenzo criteria for fault-tolerant quantum computation

1. scalable physical system with well-characterized qubits
2. ability to initialize fiducial state
3. decoherence times $\gg$ gate operation time
4. qubit-specific measurement capability
5. universal set of quantum gates

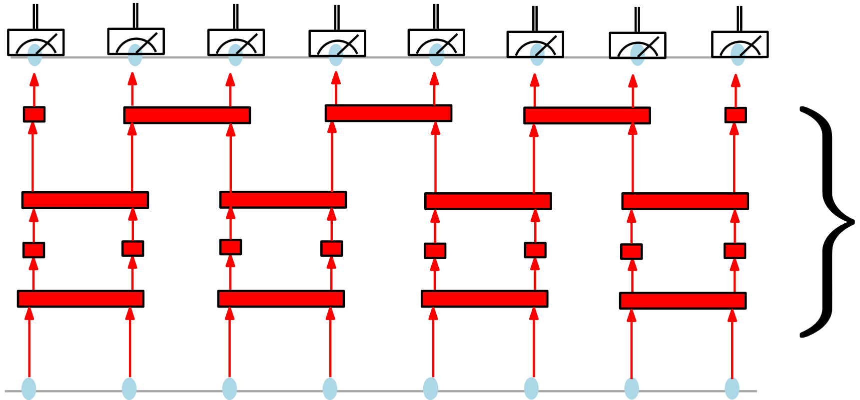
$|0\rangle$



David DiVincenzo

measurement

circuit



depth

time

initialization $|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots \quad \cdots \otimes |0\rangle \otimes |0\rangle$ k *logical* qubits

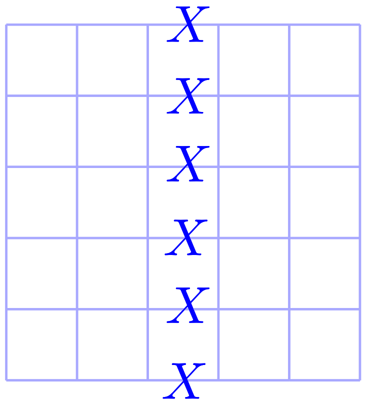
The code space of Kitaev’s toric code

Logical operators in Kitaev's toric code

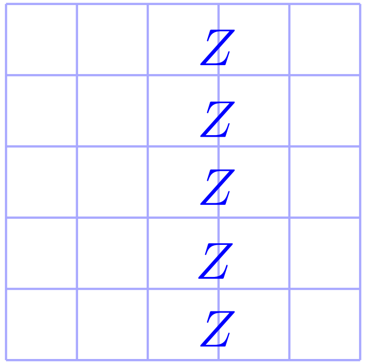
The operators $\overline{X}_1, \overline{Z}_1, \overline{X}_2, \overline{Z}_2$

- preserve the code space $\mathcal{L}$, i.e., are *logical*
- satisfy Pauli commutation relations

$\overline{X}_1 =$



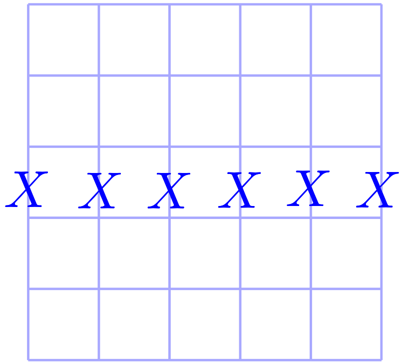
$\overline{Z}_2 =$



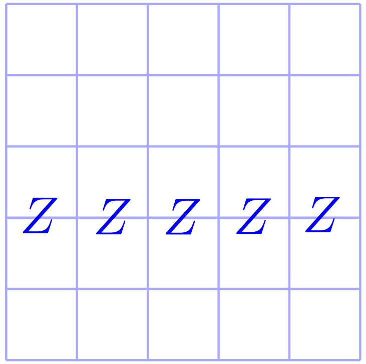
$\Rightarrow$ They define a factorization of the code space $\mathcal{L} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ such that

$\overline{X}_1 \cong X \otimes I$
$\overline{Z}_1 \cong Z \otimes I$
$\overline{X}_1 \cong X \otimes I$
$\overline{X}_2 \cong I \otimes X$

$\overline{X}_2 =$



$\overline{Z}_1 =$



Logical operators in Kitaev's toric code: commuting subalgebras

these 4 commute:

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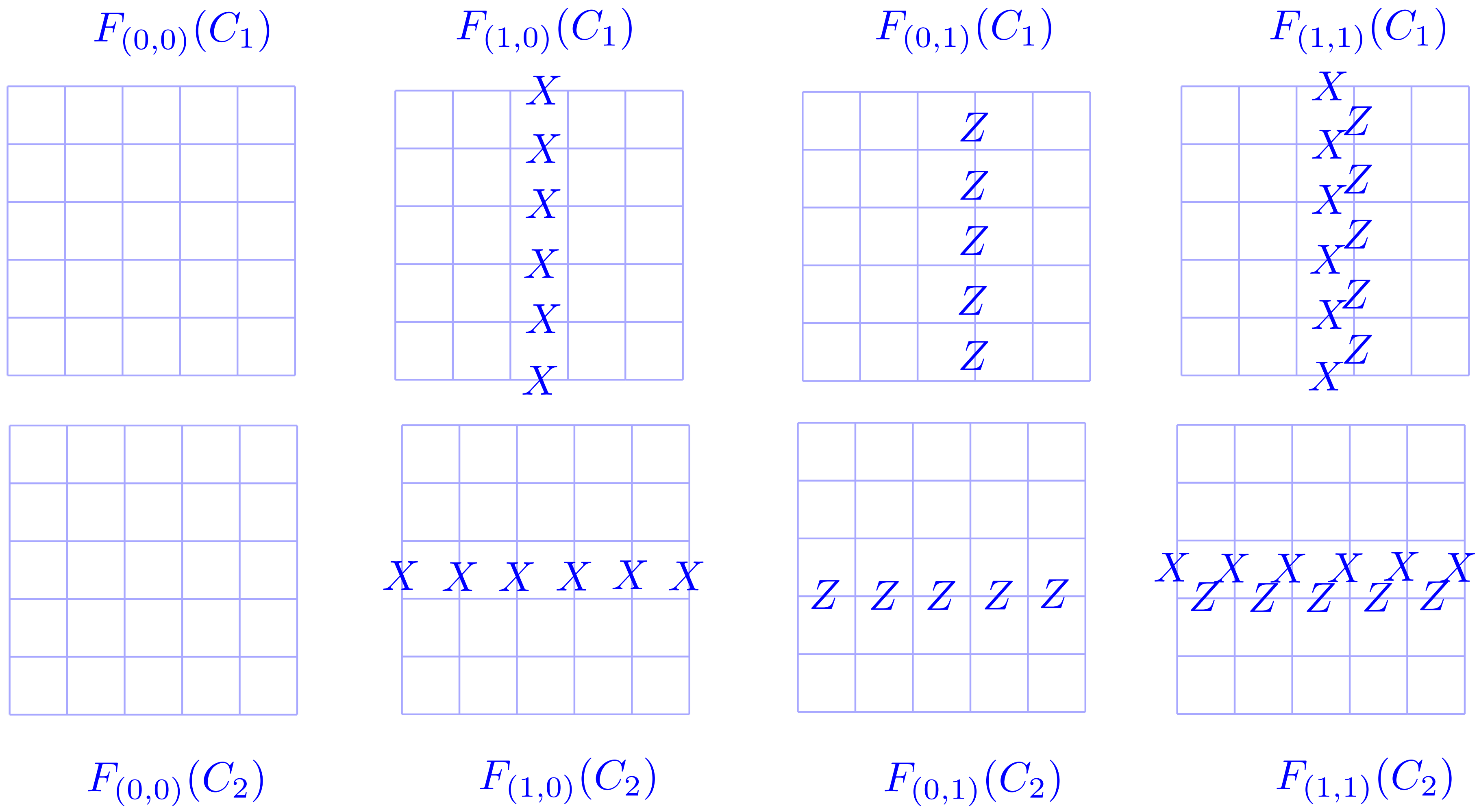
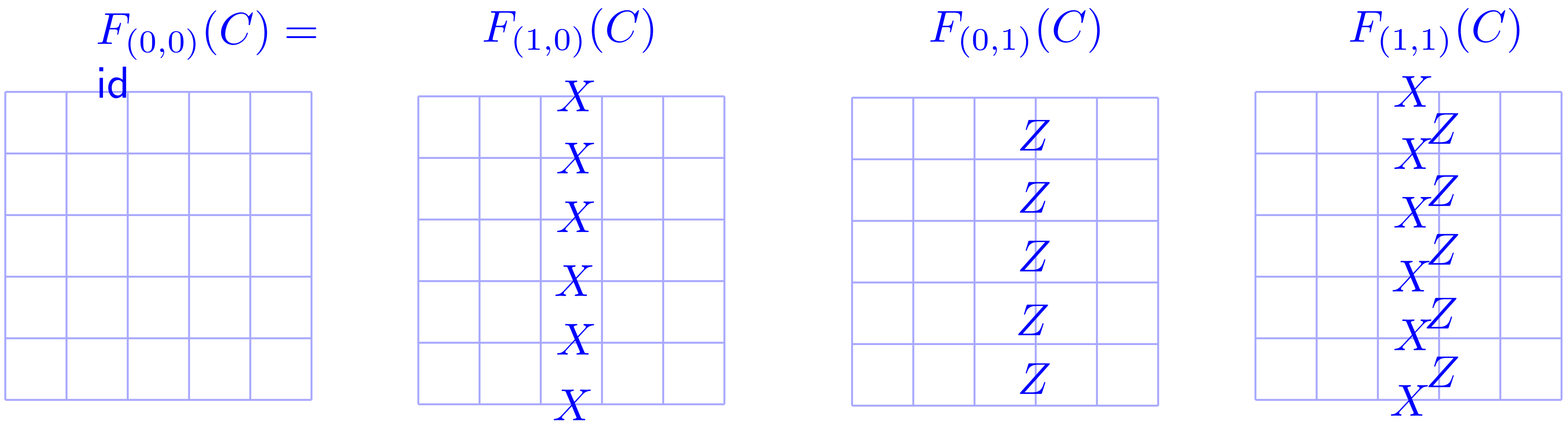


``Flux''-basis states associated with loops on a torus

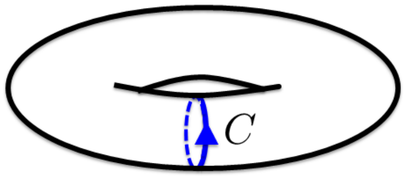
these 4 commute:



we can use the following 4 orthogonal projections to label basis states of the code space:

(These are the idempotents of the **Verlinde algebra.**)

$$\begin{array}{rcll}
P_{(0,0)}(C) & = & \frac{1}{2}(\mathrm{id} + X^{\otimes L}) \cdot \frac{1}{2}(\mathrm{id} + Z^{\otimes L}) & |1\rangle_C \\
P_{(1,0)}(C) & = & \frac{1}{2}(\mathrm{id} - X^{\otimes L}) \cdot \frac{1}{2}(\mathrm{id} + Z^{\otimes L}) & |e\rangle_C \\
P_{(0,1)}(C) & = & \frac{1}{2}(\mathrm{id} + X^{\otimes L}) \cdot \frac{1}{2}(\mathrm{id} - Z^{\otimes L}) & |m\rangle_C \\
P_{(1,1)}(C) & = & \frac{1}{2}(\mathrm{id} - X^{\otimes L}) \cdot \frac{1}{2}(\mathrm{id} - Z^{\otimes L}) & |\epsilon\rangle_C
\end{array}$$



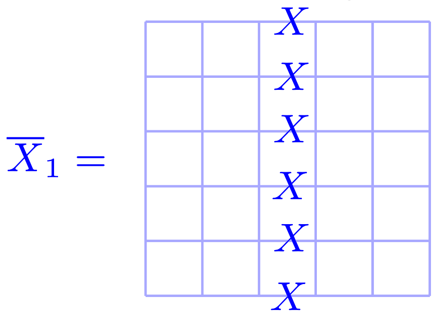
Every non-contractible closed loop C gives rise to a basis $\mathcal{B}_C$ of the code space

Fault-tolerant gates (on Kitaev’s toric code)

Fault-tolerant execution logical gates: three ways

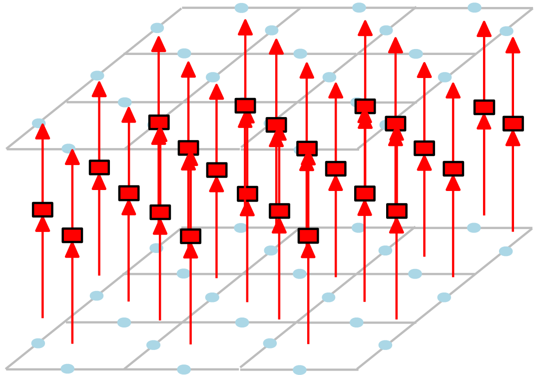
1) Apply a string-operator

- only gives logical Pauli operators
- does not generalize



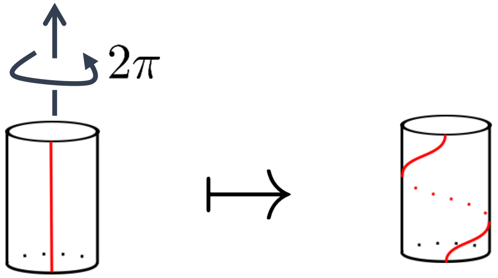
2) Apply a short (transversal) quantum circuit

- gives certain Clifford operations
- generalization?



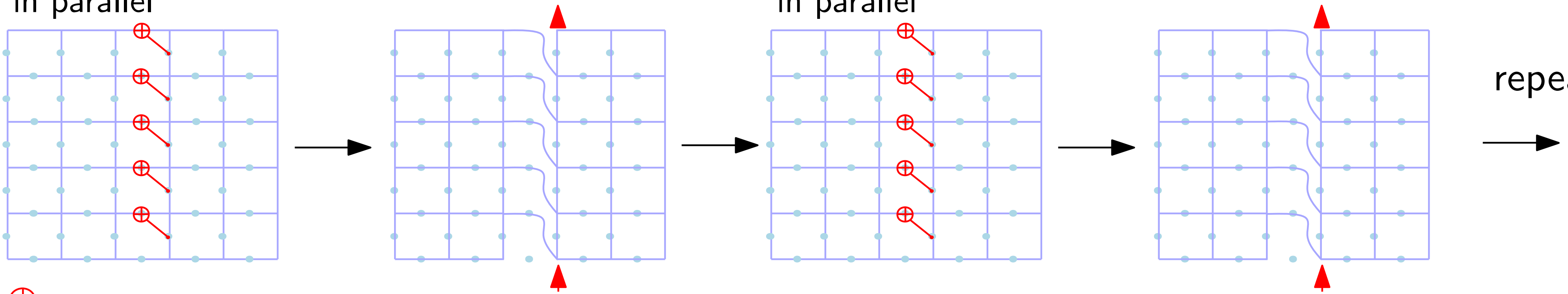
3) Apply code deformation (sequence of codes)

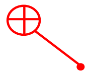
- generalizes to other models: mapping class group representation
- gives universal gate sets (in certain models)!

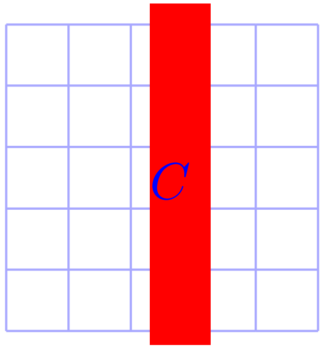


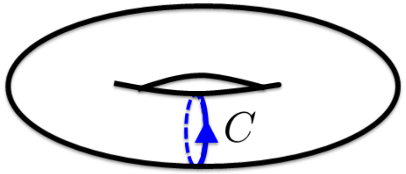
Mapping class group representation and toric code

apply L CNOTs in parallel → relocate qubits → apply L CNOTs in parallel → relocate qubits → repeat L times →



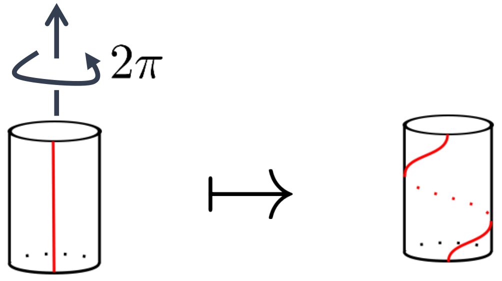

CNOT gate



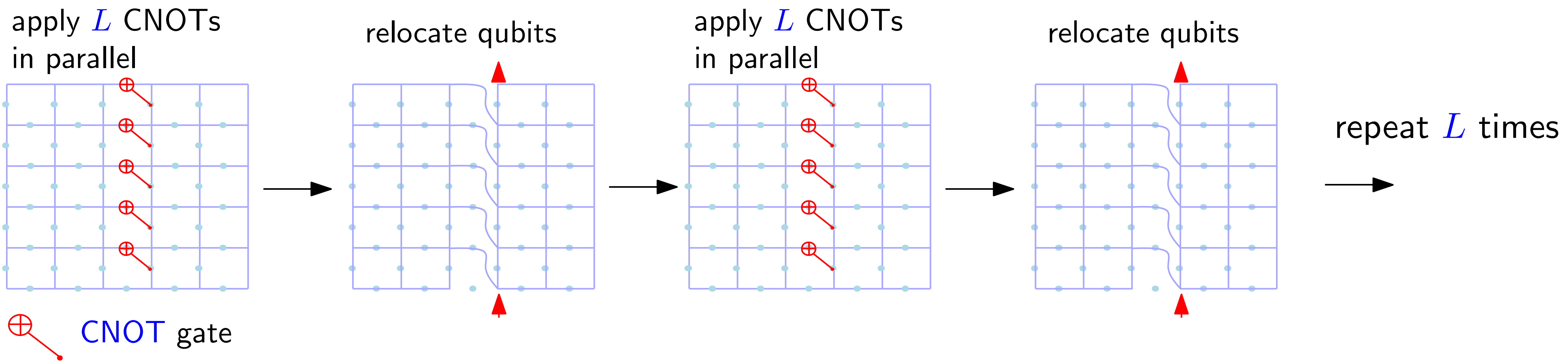


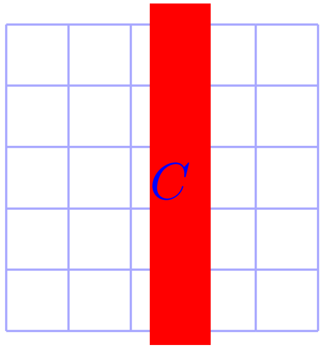
basis states of the code space:

$\vert 1\rangle_C$	are **eigenvectors**	1
$\vert e\rangle_C$	**of this operation**	-1
$\vert m\rangle_C$	with eigenvalues	-1
$\vert \epsilon\rangle_C$		1

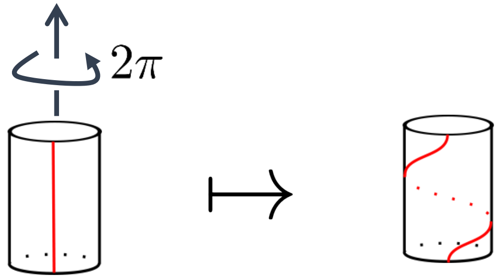


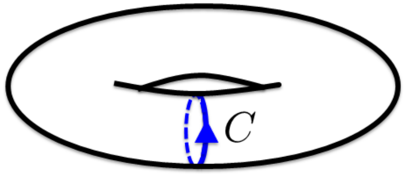
Mapping class group representation and toric code





$\Rightarrow$ For every closed, no-contractible loop C, there is a logical gate $U(C)$ implementable in depth L





Each C defines an element $\vartheta_C \in$ MCG of the mapping class group of the torus (twisting along C).

$\vartheta_C \mapsto U(C)$ gives a (projective) representation of MCG

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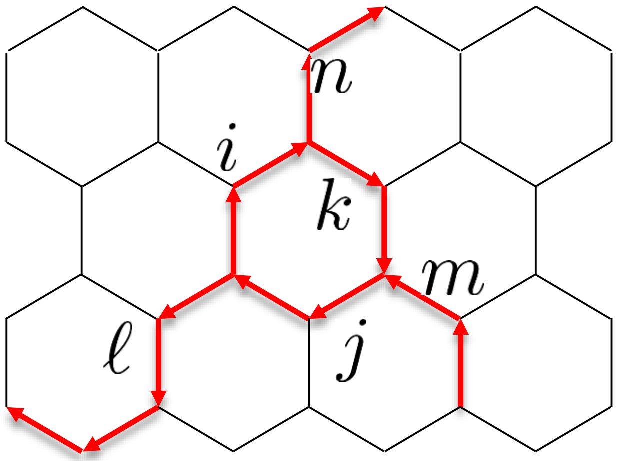
- Our work: The **Turaev-Viro code**

  - relationship to 3-manifold invariants
  - ground states
  - mapping class group representations
  - protected gates

The Levin-Wen/Turaev-Viro code

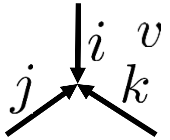
ingredients:

- finite set of "particle labels"
- involution operation on particle labels
- set of allowed triples
- scalars and a tensor



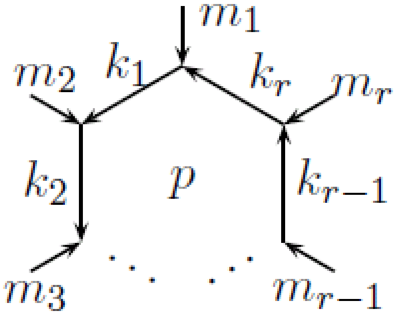
local Hilbert space $\mathbb{C}^d$
associated to every edge

vertex operator: $A_v = \sum_{(i,j,k)\ \text{allowed}} |ijk\rangle\langle ijk|$



plaquette operator: $B_p = \frac{1}{\mathcal{D}^2} \sum_{\vec{k},\vec{k}',\vec{m}} \sum_i d_i \left(\prod_{t=1}^{r} F^{m_t k_t^* k_{t-1}}_{i k'_{t-1} (k'_t)^*} \right) |\vec{k}',\vec{m}\rangle\langle\vec{k},\vec{m}|$

$|\vec{k},\vec{m}\rangle =$



Code space $\mathcal{L} \subset (\mathbb{C}^d)^{\otimes N}$

$$\mathcal{L} = \{|\Psi\rangle \mid B_p|\Psi\rangle = |\Psi\rangle\ \forall p, A_v|\Psi\rangle = |\Psi\rangle\ \forall v\}$$

Levin & Wen, Phys.Rev. B71 (2005) 045110

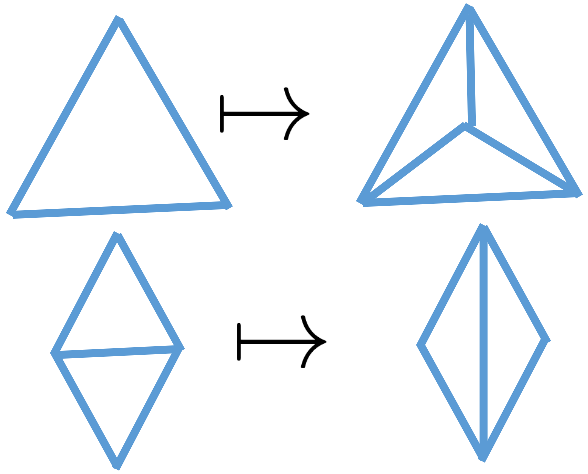
Manifold-invariants from triangulations

Consider **closed n-manifolds modulo homeomorphism**

FACT: For n=2,3, every equivalence class has a triangulated representative.

FACT (Pachner): n-manifolds homeomorphic $\Longleftrightarrow$
triangulations related sequence of Pachner moves.

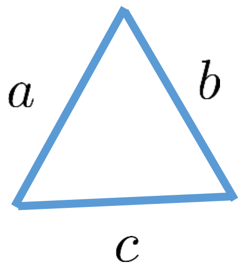
Pachner moves: finite list of local changes of triangulation, e.g., in n=2:



Recipe for constructing invariants:

- associate scalar to every triangulation
- show invariance under Pachner moves

Example: State-sum invariants



$\mapsto \quad F_{abc}$

associate scalar with (colored) triangle

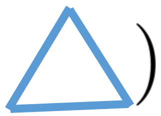
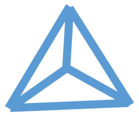
define invariant by summing over edge colorings:

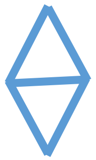
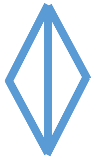
$$I(M) = \mathcal{D}^{-\#\text{triangles}} \sum_{\phi} \prod_{\text{triangles } t} g_t^{\phi}$$

triangulated 2-manifold

sum over all colorings

Compatibility with Pachner moves

$I($  $) = I($  $)$

$I($  $) = I($  $)$

is equivalent to algebraic conditions

$$\mathcal{D}^{-1} F_{abc} = \mathcal{D}^{-3} \sum_{x,y,z} F_{axz} F_{xby} F_{zyc}$$

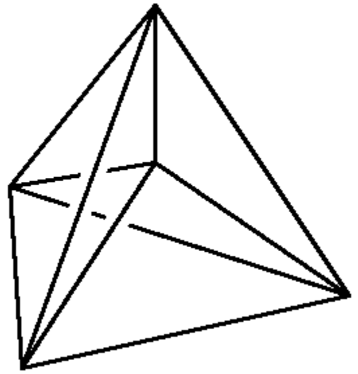
$$\sum_x F_{abx} F_{cxd} = \sum_y F_{ayc} F_{dyb}$$

The Turaev-Viro 3-manifold invariant

3-manifold

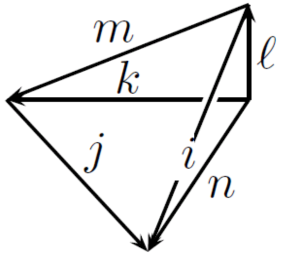
(closed)

triangulate →



sum over colorings →

TV-invariant



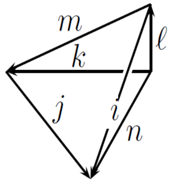
$$\mapsto \frac{F^{i^* jm}_{k\ell^* n}}{\sqrt{d_m d_n}}$$

scalar associated with (colored) tetrahedron

$$\mathsf{TV}(M) = \mathcal{D}^{-2|V_M|} \sum_{\text{colorings } \phi} \prod_{\text{edges } e} d_{\phi(e)} \prod_{\text{tetrahedra} t} g_t^{\phi}$$

sum over all ``allowed'' colorings

Algebraic conditions for invariance
(via Pachner moves)



$$\mapsto \frac{F^{i^* jm}_{k\ell^* n}}{\sqrt{d_m d_n}}$$

$$\mathrm{TV}_{\mathcal{C}}(M) = \mathcal{D}^{-2|V_M|} \sum_{\text{colorings } \phi} \prod_{\text{edges } e} d_{\phi(e)} \prod_{\text{tetrahedra} t} g_t^{\phi}$$

If

$$1 = 1^*$$
$$d_1 = 1$$
$$d_i = d_{i^*}$$
$$\mathcal{D} = \sqrt{\textstyle\sum_i d_i^2}$$
$$d_i d_j = \sum_k \delta_{ijk} d_k$$
$$\sum_m \delta_{ijm^*}\delta_{mkl^*} = \sum_m \delta_{jkm^*}\delta_{iml^*}$$
$$F^{ijm}_{k\ell n}\delta_{ijm}\delta_{k\ell m^*} = F^{ijm}_{k\ell n}\delta_{i\ell n}\delta_{jkn^*}$$
$$\textstyle\sum_n F^{m\ell q}_{kpn} F^{jip^*}_{mns} F^{jsn}_{\ell kr} = F^{jip^*}_{q^* kr} F^{r^* iq^*}_{m\ell s}$$
$$(F^{ijm}_{k\ell n})^* = F^{i^* j^* m^*}_{k^* \ell^* n^*}$$
$$F^{ijm}_{k\ell n} = F^{jim}_{\ell kn^*} = F^{\ell km^*}_{jin} = F^{imj}_{k^* n\ell}\sqrt{\frac{d_m d_n}{d_j d_\ell}}$$
$$F^{ii^* 1}_{j^* jk} = \sqrt{\frac{d_k}{d_i d_j}}\delta_{ijk}$$

*: involution on set of colors
1: special color

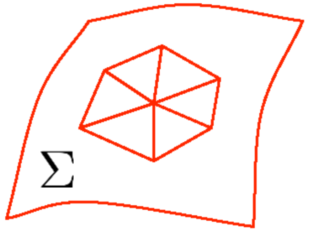
$$\delta_{ijk} \in \mathbb{N}\cup\{0\}$$
$$F^{ijm}_{k\ell n} \in \mathbb{R}$$
$$d_i \in \mathbb{R}_{>0}$$

then $\mathrm{TV}_{\mathcal{C}}$ is a 3-manifold invariant

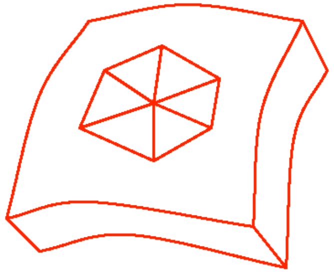
A *spherical category* $\mathcal{C}$ is/provides a solution to these equations.

(Barrett and Westbury, hep-th/9311155)

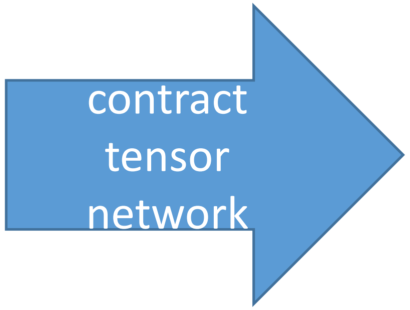
The Turaev-Viro code $\subset (\mathbb{C}^d)^{\otimes |E|} \cong$ edge colorings of surface triangulation



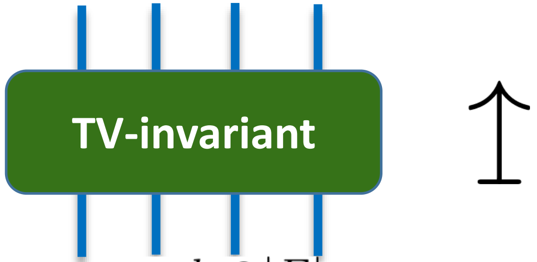
$\Sigma \times [-1, 1]$



(extend triangulation from $\Sigma \times \{\pm 1\}$)



$(\mathbb{C}^d)^{\otimes |E|}$

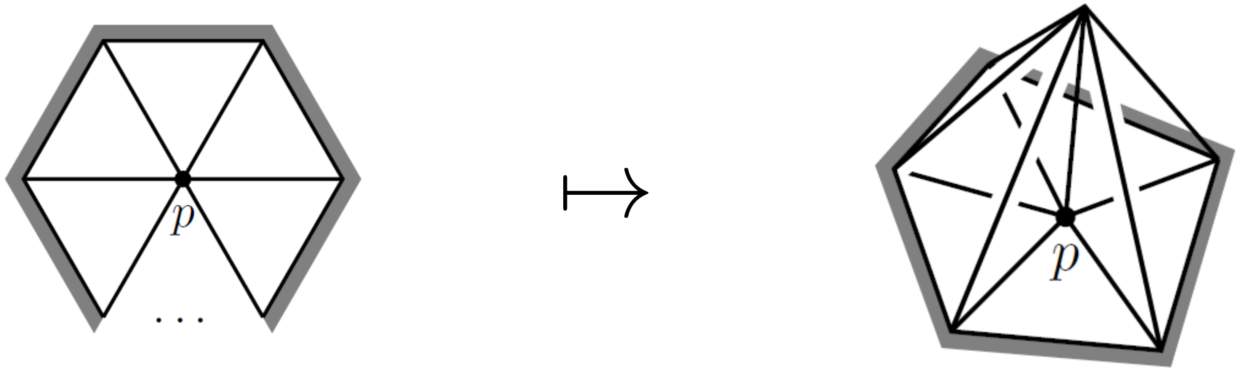


$(\mathbb{C}^d)^{\otimes |E|}$

Turaev-Viro code: support of this projection in the Hilbert space $(\mathbb{C}^d)^{\otimes |E|}$

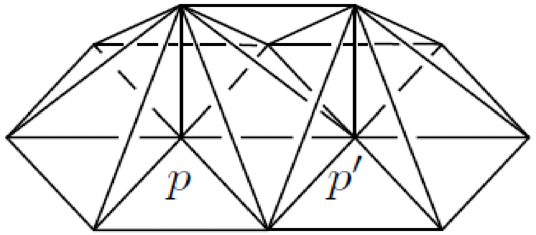
Local stabilizers: attaching blisters - set of local operators which are

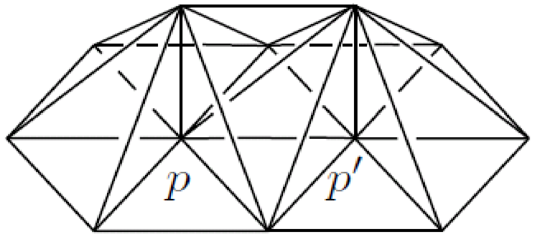
- projections
- mutually commuting
- stabilize code space



Blisters: properties from (manifold)invariance

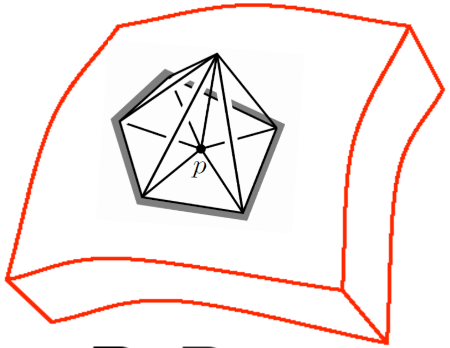
commuting:

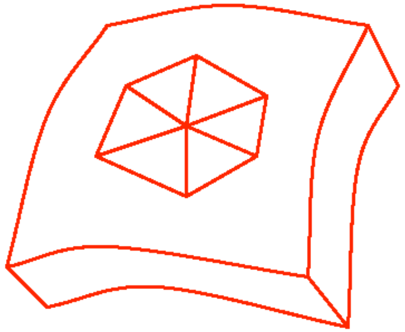




$$B_p B_{p'} = B_{p'} B_p$$

stabilize code space:

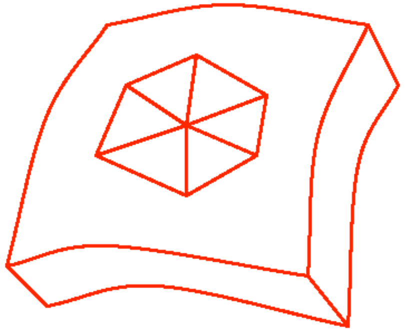




$$B_p P_{\mathrm{TV}} = P_{\mathrm{TV}}$$

project onto code space

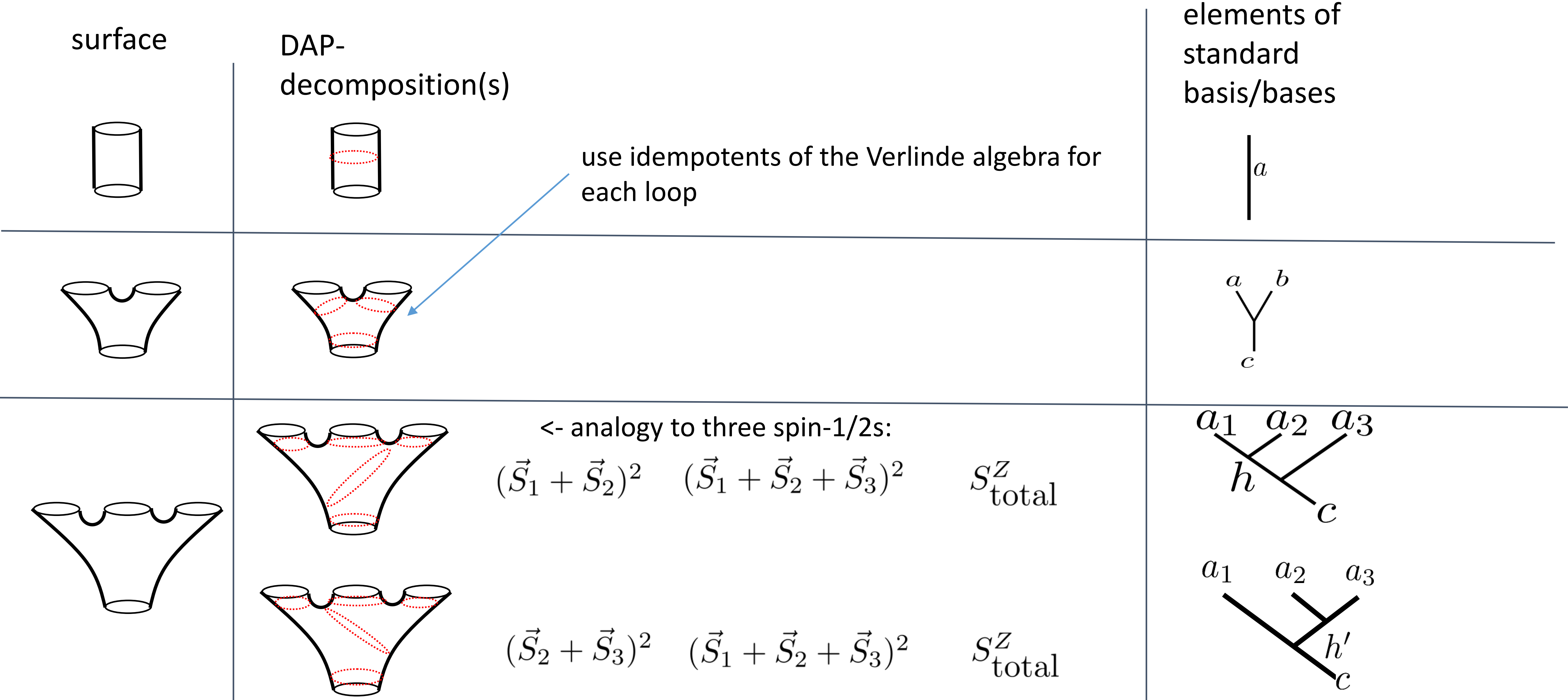
$$\prod_p B_p =$$



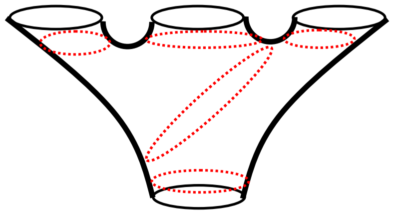
The code space of the Turaev-Viro code

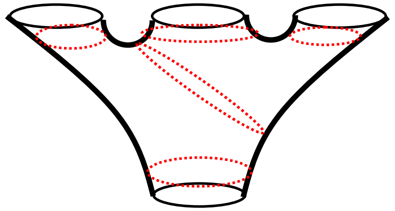
``Standard bases’’ from maximal sets of commuting observables

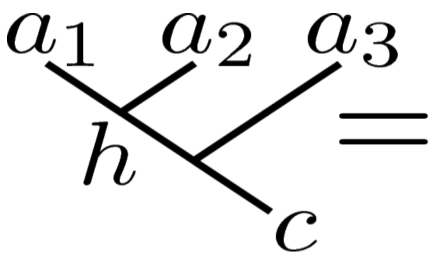
Any DAP-decomposition correspond to a “complete set of observables” and defines a basis of the code space.



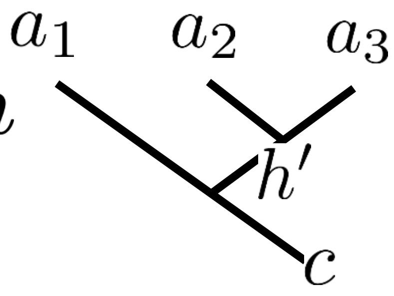
F-move: basis change between bases associated with different DAP-decompositions







$$= \sum_{h'} F^{a_2 a_1 h}_{c a_3 h'}$$

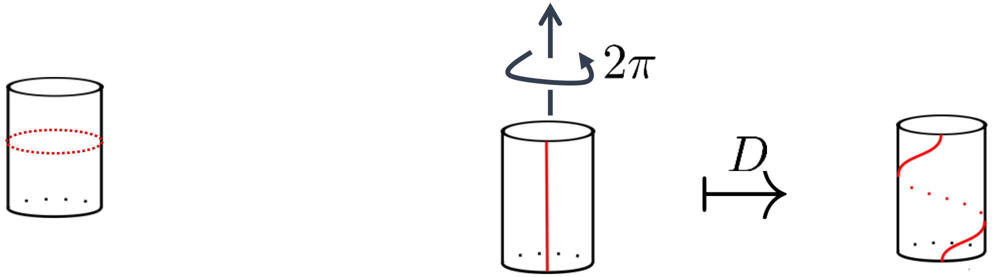


some (controlled) unitary $U(a_2, a_1, a_3, c)_{h,h'}$

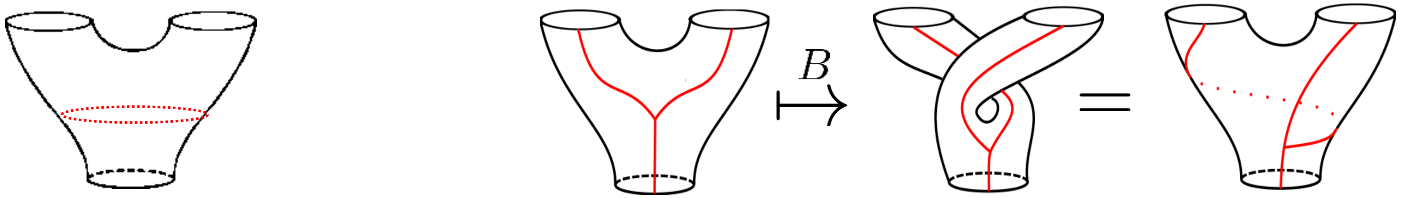
....analogous to spin-1/2- 6j symbols

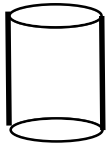
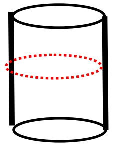
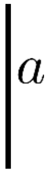
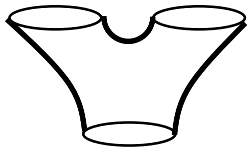
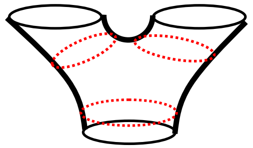
Mapping class group (generators) and basis elements

Dehn-twist:

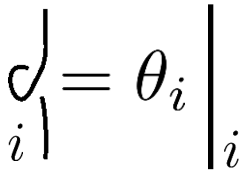


Braid-move:

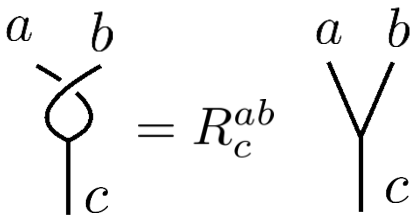


surface	DAP-decomposition(s)	elements of standard basis/bases
		
		

topological phase



R-matrix



Note: this is just a fancy way of writing equation

$$D|i\rangle = \theta_i|i\rangle$$

D = twist

$$B|b,a;c\rangle = R_c^{ab}|a,b;c\rangle$$

B = braid

Conditions for MCG-representations:

(Moore and Seiberg)

•Consistency of basis changes:

$$\sum_n F^{m\ell q}_{kpn} F^{jip^*}_{mns} F^{jsn}_{\ell kr} = F^{jip^*}_{q^* kr} F^{r^* iq^*}_{m\ell s}$$

(pentagon-identity)

spherical

•Compatibility of basis changes with action of braiding generators:

$$R^{ki}_m F^{k^* i^* m}_{\ell j^* g} R^{kj}_g = \sum_n F^{i^* k^* m}_{\ell j^* n} R^{kn}_\ell F^{j^* i^* n}_{\ell k^* g}$$

$$\theta_i = (R^{i^* i}_1)^*$$

(hexagon-identity)

braided

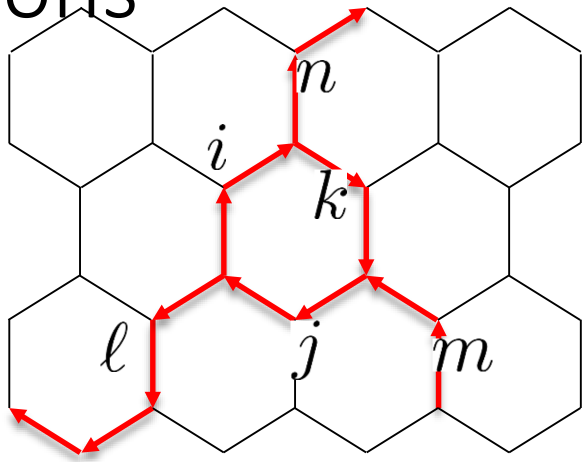
• unitarity of representation:

....

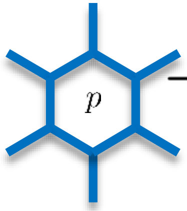
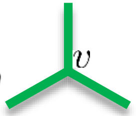
modular category

Basis states for the Turaev-Viro code

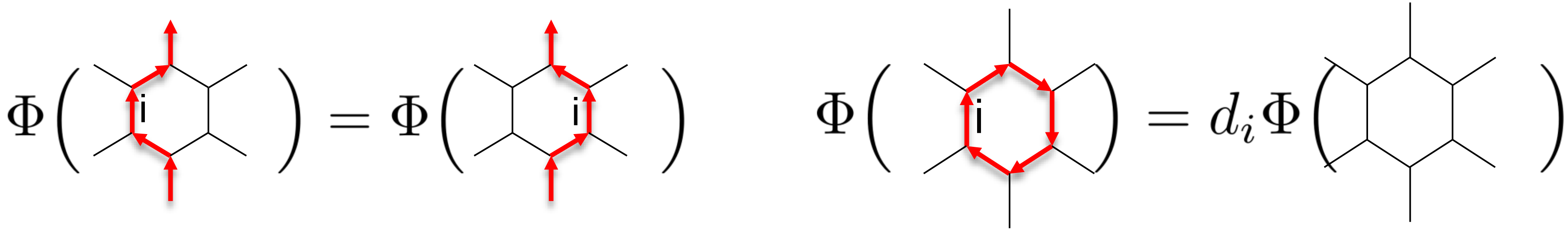
Levin-Wen ground space and local relations



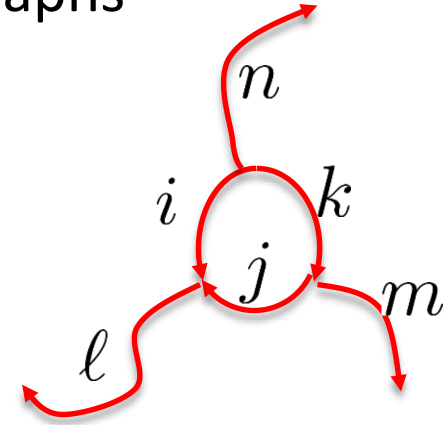
qudit lattice Hamiltonian

$H = -\sum_p$  $-\sum_v$ 

ground state coefficients in computational basis satisfy discrete local “skein” relations, e.g.,



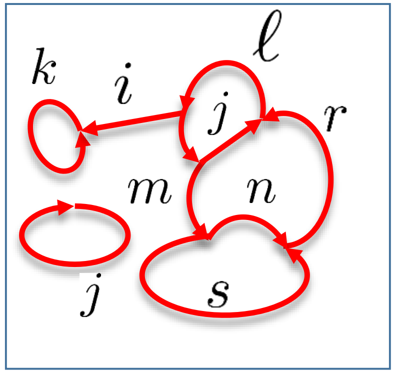
Consequence: Ground space is isomorphic to Hilbert space of ribbon graphs (“pictures”) modulo local equivalence relations



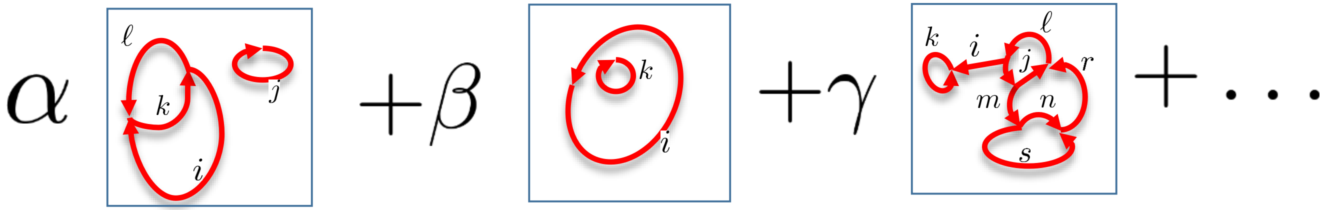
ribbon graph space $\mathcal{H}_\Sigma$

Ribbon graphs Hilbert space $\mathcal{H}_\Sigma$ for general category

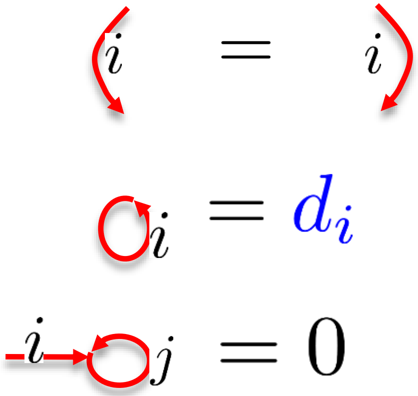
trivalent *labeled directed* graphs (with loops) embedded in Σ



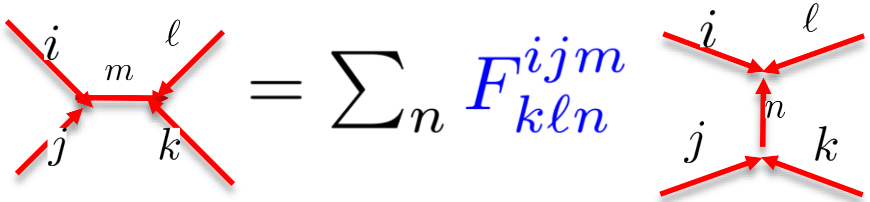
State: formal linear combination of ribbon graphs



modulo **local relations**

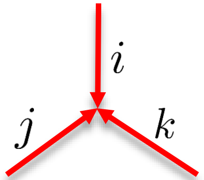


q-dimensions

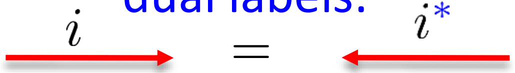


F-symbol

fusion rules
(set of allowed triples):

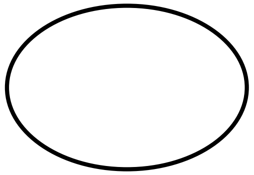
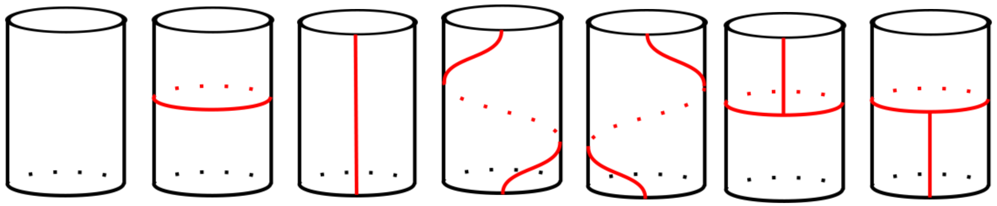
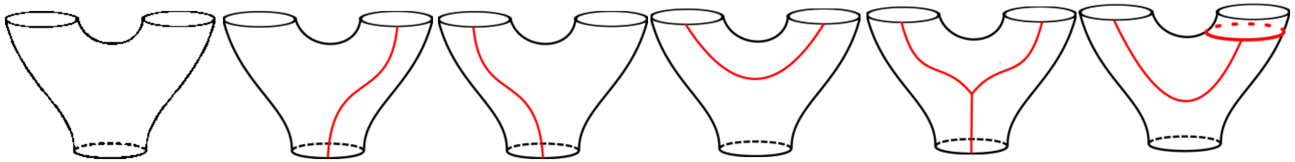


dual labels:



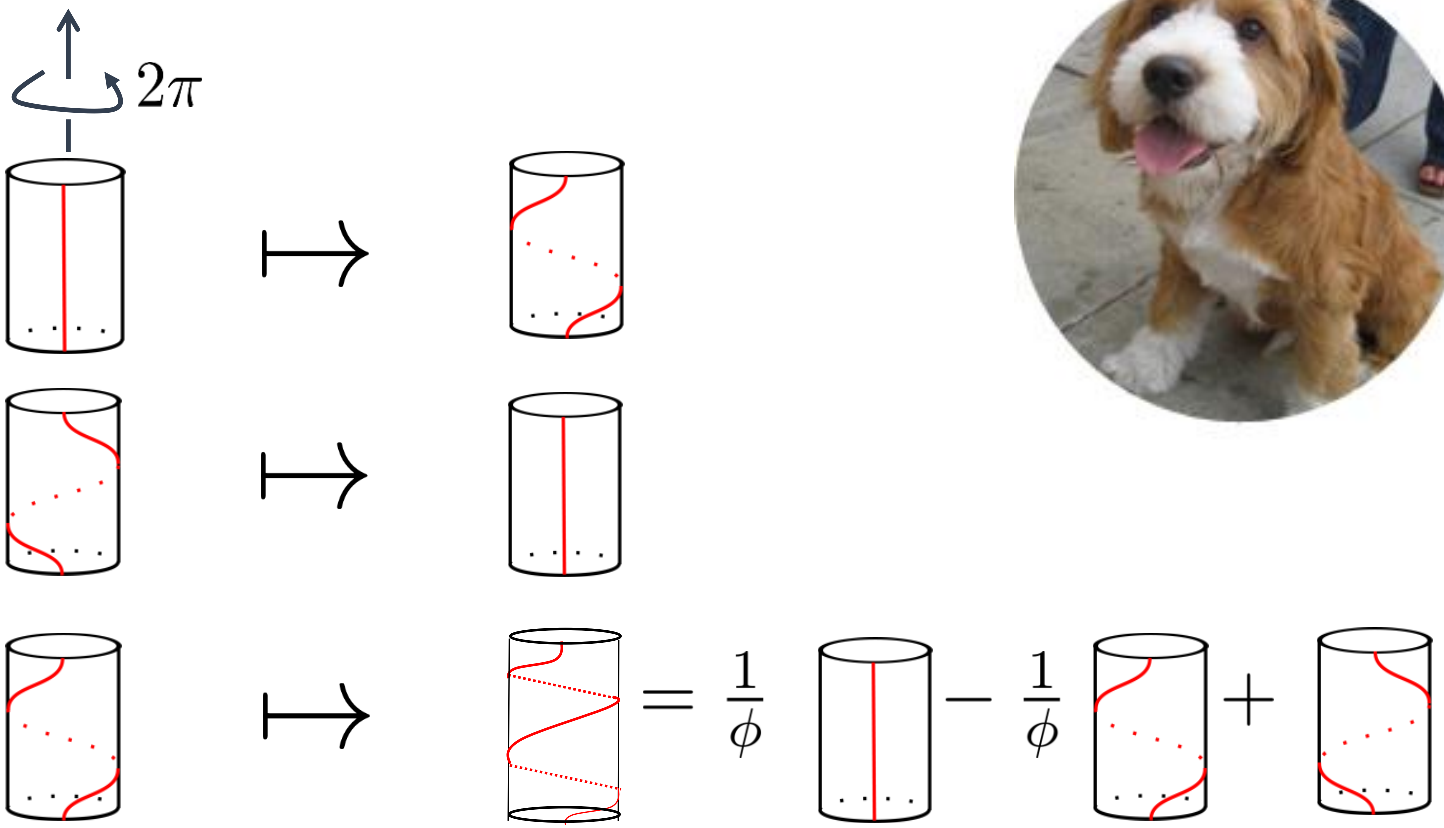
trivial label (absence of string):

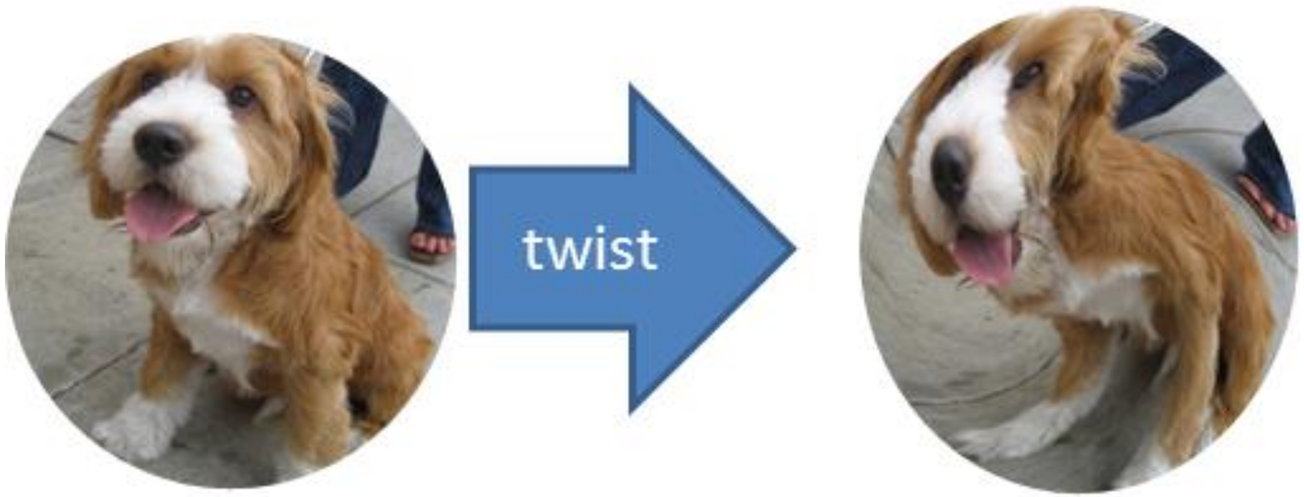
Ribbon graph bases of $\mathcal{H}_\Sigma$ for $\color{blue}{Fib}$

Surface Σ	$dim \mathcal{H}_\Sigma$	Example basis
Disc (1-punctured sphere)	**1**	
Annulus (2-punctured sphere)	**7**	
Pair of pants (3-punctured sphere)	**65**	 …
n-punctured sphere	$2^{\Omega(n)}$	

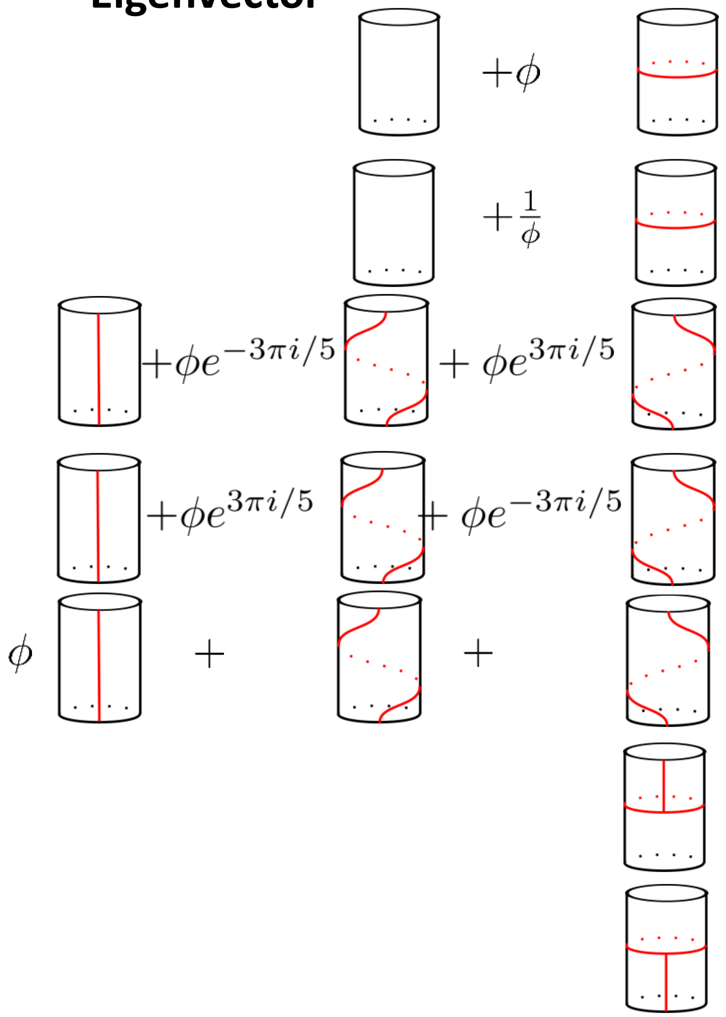
Next: Description of bases compatible with action of (generators of) mapping class group!

Action of Dehn twist on $\mathcal{H}_{\Sigma_2}$ for Fib





Goal: identify "**fusion tree basis**" (eigenvectors of twist)

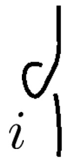
Eigenvector	eigenvalue (twist)	name	boundary labels
	1	$1 \otimes 1$	$1, 1$
	1	$\tau \otimes \tau$	
	$e^{4\pi i/5}$	$1 \otimes \tau$	τ, τ
	$e^{-4\pi i/5}$	$\tau \otimes 1$	
	1	$\tau \otimes \tau$	
	1	$\tau \otimes \tau$	$\tau, 1$
	1	$\tau \otimes \tau$	$1, \tau$

Anyonic fusion basis obtained by diagonalization

fusion space basis element

$$V_i^i = \mathbb{C} \Big| i$$

topological phase



$$= \theta_i \Big| i$$

anyon type

i

of "doubled" theory

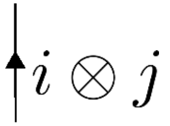
$Fib \otimes Fib'$

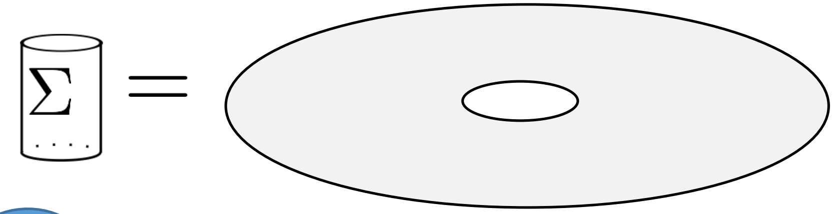
multiplicity index for different realizations as subspaces of $\mathcal{H}_{\Sigma_2}$

Anyonic fusion basis from "doubled" manifold $\Sigma \times [-1,1]$

Goal: find anyonic fusion basis states on Σ

Example: find element for annulus

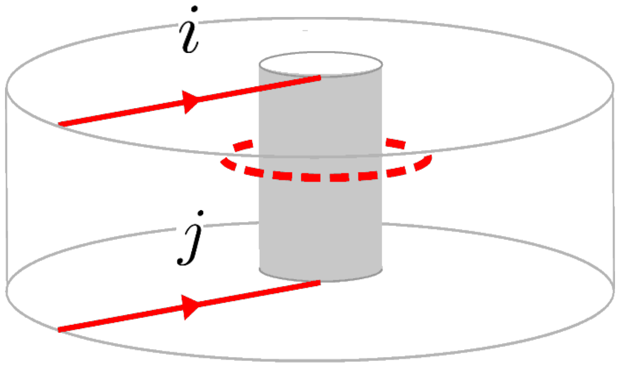


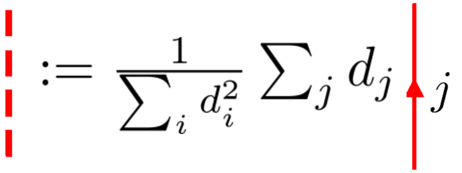


Intermediate step: identify relevant ribbon graphs on $\Sigma \times [-1,1]$

A recipe which does not involve diagonalization

some ribbon graph on

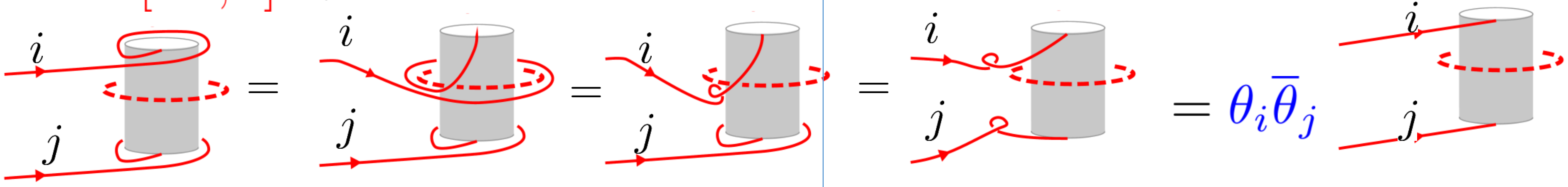


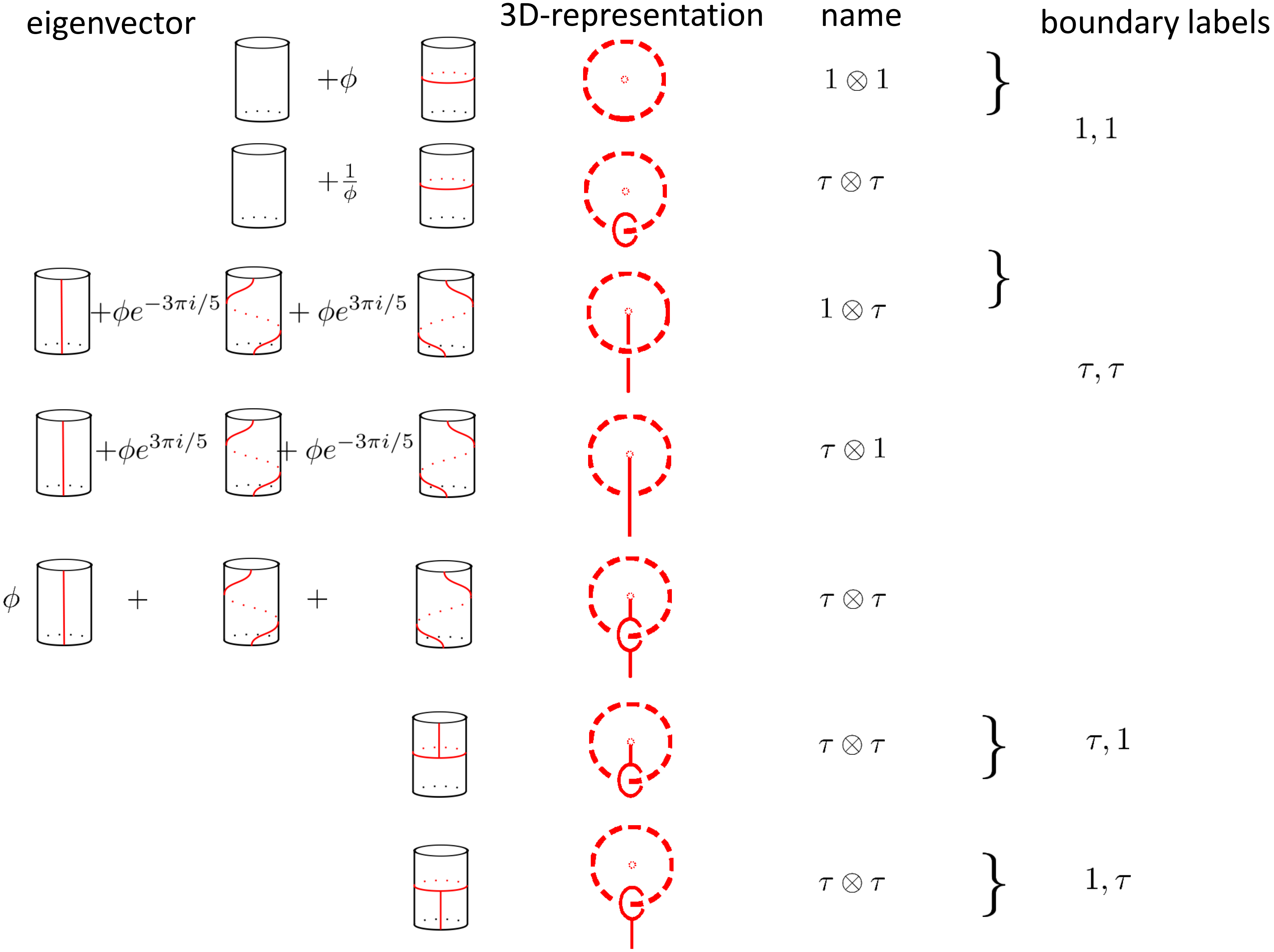


simple derivation of topological phase:

Map ribbon graphs $\Sigma \times [-1,1] \to \Sigma$

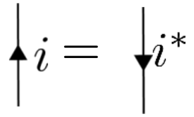
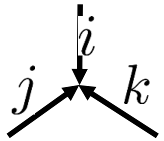
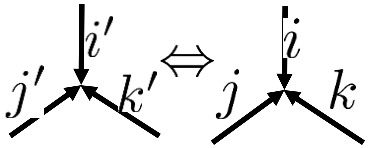
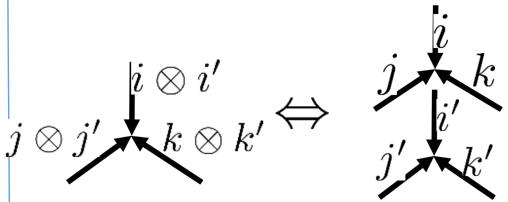
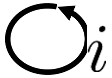
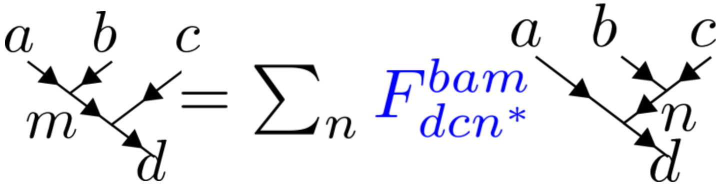
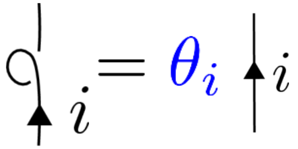
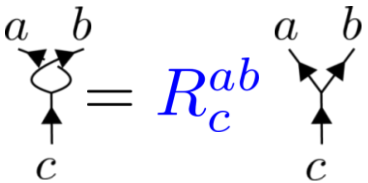
using "vacuum" lines





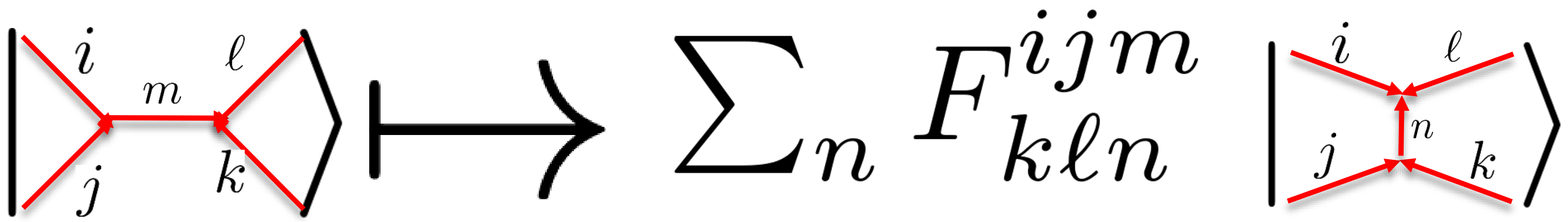
eigenvector
3D-representation
name
boundary labels
$+\phi$
$1 \otimes 1$
$1, 1$
$+\frac{1}{\phi}$
$\tau \otimes \tau$
$+\phi e^{-3\pi i/5}$
$+\phi e^{3\pi i/5}$
$1 \otimes \tau$
τ, τ
$+\phi e^{3\pi i/5}$
$+\phi e^{-3\pi i/5}$
$\tau \otimes 1$
ϕ
$+$
$+$
$\tau \otimes \tau$
$\tau \otimes \tau$
$\tau, 1$
$\tau \otimes \tau$
$1, \tau$

Derived categories: basic data

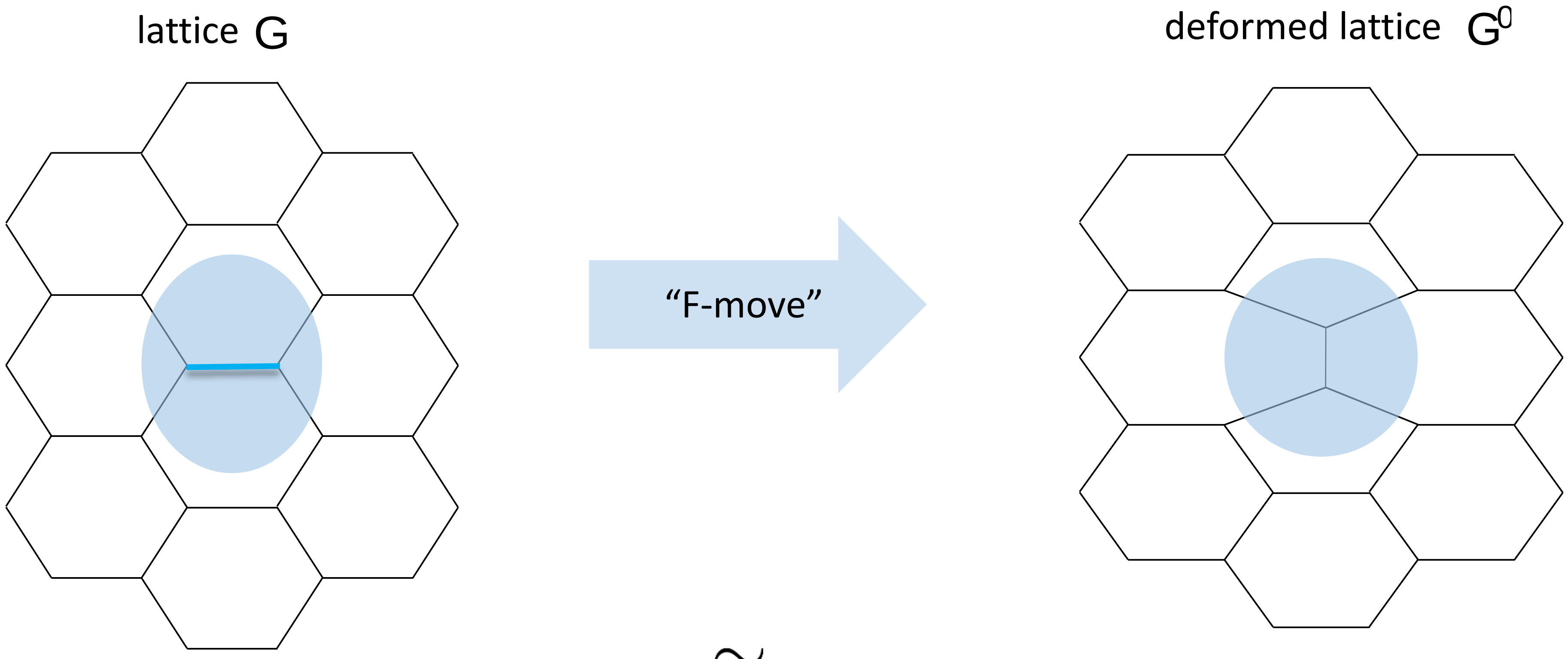
Unitary, braided, semisimple, *		modular tensor category $\mathcal{C}$	dual category $\mathcal{C}'$	doubled category $\mathcal{C}\otimes\mathcal{C}'$
	Particles	$\{1, i, j, \dots\}, *$ 	$\{i' \mid i \in \mathcal{C}\}$	$\left\{ i \otimes j' \mid \begin{smallmatrix} i\in\mathcal{C}, \\ j'\in\mathcal{C}' \end{smallmatrix} \right\}$
	Fusion rules	 (set of) allowed triples		
	q-dim	 $= d_i$	$d_{i'} = d_i$	$d_{i\otimes j'} = d_i d_{j'}$
	F-matrix		$F^{b'a'm'}_{d'c'n'^*} = F^{bam}_{dcn^*}$	$F \otimes F'$
	top. phase		$\theta_{i'} = \bar{\theta}_i$	$\theta_{i\otimes j'} = \theta_i \theta_{j'}$
	R-matrix		$R^{a'b'}_{c'} = \overline{R^{ab}_c}$	$R \otimes R'$

Computation with Turaev-Viro codes

Different lattices and F-move isomorphism

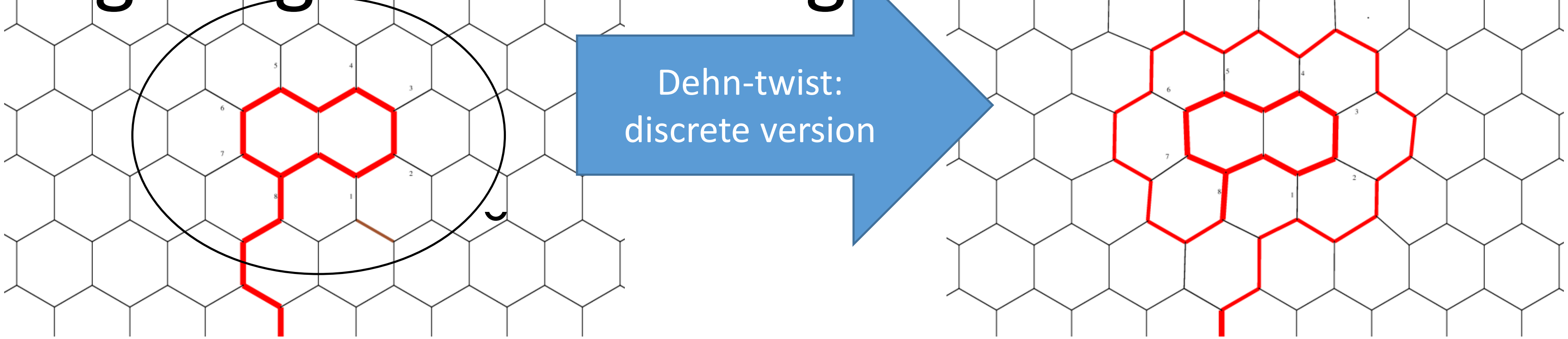


For unitary tensor categories, this is a **unitary 5-qudit gate.**



$\mathcal{H}_\mathcal{G}$ ground space of spins on original lattice $\xrightarrow{\cong}$ $\mathcal{H}_{\mathcal{G}'}$ ground space of spins on deformed lattice

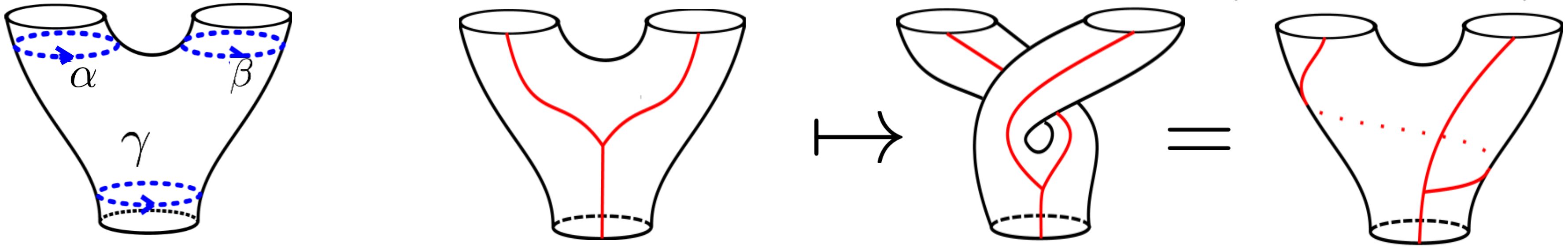
Logical gates: Executing braids



Can be implemented by sequence of $O(|\gamma|^2)$ F-moves (5-qudit gates)

π -twists can be implemented similarly, therefore braids:

$$B = \sqrt{D_\gamma}\sqrt{D_\alpha^{-1}}\sqrt{D_\beta}$$

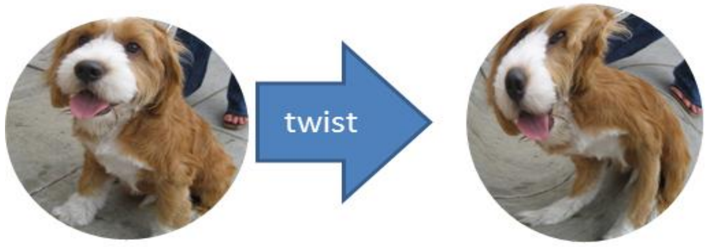


universal gate set:

- braids generate dense subgroup of unitaries on subspace of $\mathcal{H}_\Sigma$ for (doubled) Fib
- for approriate encoding, approximation of universal gate set by Solovay-Kitaev

(Freedman, Larsen, Wang'02)

Gate sets obtained from the mapping class group



TQFT	*mapping class group (braiding) contained in*
$D(\mathbb{Z}_2)$	Pauli group
abelian anyon model	generalized Pauli group
Fibonacci model	universal
Ising model	Clifford group
generic anyon model	model-dependent
generic anyon model	universal

Conclusions and open problems

- Turaev-Viro codes offer a rich class of examples for potential platforms for topological quantum computation.
- The mapping class group representation can be "decomposed" using the string-net formalism
- Explicit constructions of protected/transversal gates for TQFTs?
- Performing syndrome-measurement & error correction, thresholds for fault-tolerance?
- Higher-dimensional generalizations?