Dynamical vector fields on the manifold of quantum states

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To describe any physical system we need to identify:

- *States*
- *Observables*
- *Probability function*
- *Evolution equation*
- *Composition rule of systems*

Classical systems

A symplectic manifold $(\mathcal{M}, \omega)$

States: probability distibutions or probability measures, Liouville measure, comparison measure

Observables: $\mathcal{F}(\mathcal{M}, \mathbb{R})$

Probability function: for any Borelian $\mathcal{B} \subseteq \mathbb{R}$, ρ a state, f observable:

$$\int_{f^{-1}(\mathcal{B})} \rho \, \mathrm{d}\mu_L =$$ Probability to find a value of f in $\mathcal{B}$ when the system is in the state ρ

Evolution: $i_\Gamma \omega = -\mathrm{d}H \qquad \frac{\mathrm{d}}{\mathrm{d}t} f = \{H, f\} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \rho = \{\rho, H\}$

Composition: $(\mathcal{M}_1 \times \mathcal{M}_2, \omega_1 \oplus \omega_2)$

Quantum systems

(Schrödinger picture) "wave mechanics": $\mathcal{H}$ *complex separable Hilbert space*

(Pure) States: *rays in* $\mathcal{H}$, $\mathcal{PH}$ *complex projective space, Hilbert manifold*

$$\rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

Observables: *real elements in* $\mathcal{B}(\mathcal{H}) \longleftrightarrow \mathcal{H} \otimes \mathcal{H}^*$

Probability function: *for any res. of the identity* $\sum_j |e_j\rangle\langle e_j| = \mathbb{I}\,,\ \int |x\rangle \mathrm{d}x \langle x|$

and any vector $|\psi\rangle\,,\ p_j(\psi) = \dfrac{\langle e_j|\psi\rangle\langle\psi|e_j\rangle}{\langle\psi|\psi\rangle}\,, p_j \geq 0\,, \sum_j p_j = 1$

similarly $\psi^*(x)\psi(x) = \dfrac{\langle x|\psi\rangle\langle\psi|x\rangle}{\langle\psi|\psi\rangle}\,,\ \psi^*(x)\psi(x) \geq 0\,,\ \int \psi^*(x)\psi(x)\mathrm{d}x = 1$

Evolution: $\imath\hbar\dfrac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \mathbf{H}|\psi\rangle \Rightarrow \imath\hbar\dfrac{\mathrm{d}}{\mathrm{d}t}\rho_\psi = [\mathbf{H}\,, \rho_\psi]$

Composition: $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$

Quantum systems

(Heisenberg picture) "matrix mechanics": a C^*-algebra $\mathcal{A}$

Observables: *real elements in $\mathcal{A}$:* $\mathbf{A} = \mathbf{A}^\dagger$

States: *normalized, positive, linear functionals on $\mathcal{A}$*

$$\rho(\mathbb{I}) = 1\,, \quad \rho(\mathbf{A}^\dagger \mathbf{A}) \geq 0\,, \; \rho(\mathbf{A}+\mathbf{B}) = \rho(\mathbf{A}) + \rho(\mathbf{B})$$

GNS construction gives back a Hilbert space $\mathcal{H}_{GNS}$

Probability function:

$$\mathbf{E}_j^\dagger = \mathbf{E}_j \in \mathcal{A}\,, \; \sum_j \mathbf{E}_j = \mathbb{I}\,, \; \mathbf{E}_j \cdot \mathbf{E}_k = \delta_{jk} \mathbf{E}_j\,, \; \rho(\mathbf{E}_j) = p_j(\rho) \geq 0\,, \; \sum_j p_j(\rho) = 1$$

Evolution: $\imath\hbar \dfrac{\mathrm{d}}{\mathrm{d}t} \mathbf{A} = [\mathbf{A}\,, \mathbf{H}]\,, \quad \imath\hbar \dfrac{\mathrm{d}}{\mathrm{d}t} \rho = [\mathbf{H}\,, \rho]$

Composition: $\mathcal{A}_{12} = \mathcal{A}_1 \otimes \mathcal{A}_2$

Quantum systems

Other pictures:

- *Weyl-Wigner*
- *Generalized coherent states*
- *Tomographic picture*
- *Linearity versus nonlinearity*

Probabilistic-statistical interpretation of quantum mechanics.
The primary object is the space of states (we shall consider only finite-dimensional systems).

Schrödinger-Dirac picture: The Hilbert manifold of pure states.

Heisenberg, Born-Jordan: A connected closed complete and convex set $\mathcal{S}$ in some affine topological space $\mathcal{E}$.
The space of states is a stratified manifold (the boundary is not a smooth manifold) with two compatible contravariant tensor fields:

Λ Skew-symmetric, defines a Poisson bracket

$\mathcal{R}$ Symmetric, defines a Jordan algebra

Observables are real-valued functions on $\mathcal{S}$
with the following properties:

$\Lambda(\mathrm{d}f) = X_f$ *Hamiltonian vector fields with the Killing property* $L_{X_f}\mathcal{R} = 0$

$\mathcal{R}(\mathrm{d}f) = Y_f$ *Gradient vector fields*

Hamiltonian and gradient vector fields generate the tangent bundle of $\mathcal{S}$, for every stratum, and close on the Lie algebra of $SL(N, \mathbb{C})$

We find that:

- *Observables constitute a Lie-Jordan algebra;*
- *By extension to complex-valued combinations of observables we generate a C^*-algebra.*

By using a GNS construction we recover a Hilbert space.
The irriducibility requirement (a minimality condition) allows to recover the Hilbert space of the Schrödinger-Dirac picture

Example: Q-bit

States, Bloch ball in $E = \mathbb{R}^3 \subset \mathbb{R}^4$

$$\rho = \frac{1}{2}(\sigma_0 + \vec{x}\cdot\vec{\sigma}) \qquad \mathcal{S} = \{\vec{x}:\ |\vec{x}|^2 \leq 1\}$$

$$\Lambda = \frac{\epsilon_l^{jk} x^j}{2} \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^l} \qquad \mathcal{R} = \left(\delta^{jk} - x^j x^k\right) \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k}$$

Observables $\quad f = a_j x^j\,,\ a_j \in \mathbb{R}$

$$X_f = \epsilon_l^{jk} a_k x^l \frac{\partial}{\partial x^j}\,, \qquad Y_f = a_j \frac{\partial}{\partial x^j} - (a_k x^k)\left(x^j \frac{\partial}{\partial x^j}\right)$$

They generate the Lie algebra of $SL(2,\mathbb{C})$

If we consider a realization of the Lie algebra in terms of matrices we get back the complex matrix algebra generated by $\sigma_0\,,\ \sigma_1\,,\ \sigma_2\,,\ \sigma_3$

Example: Q-bit

The Lie-Jordan algebra:

$$\{x^j\,, x^k\} = \Lambda(\mathrm{d}x^j\,, \mathrm{d}x^k) = \epsilon_l^{jk} x^l$$

$$x^j \odot x^k = \mathcal{R}(\mathrm{d}x^j\,, \mathrm{d}x^k) + x^j x^k \Rightarrow x^j \odot x^j = 1\,,\; x^j \odot x^k = 0$$

We can define $x^j \star x^k = x^j \odot x^k + \imath\{x^j\,, x^k\}$ *so that:*

$$x^j \odot x^k = \frac{1}{2}\left(x^j \star x^k + x^k \star x^j\right)\;, \quad \{x^j\,, x^k\} = -\frac{\imath}{2}\left(x^j \star x^k - x^k \star x^j\right)$$

Example: Q-bit

Remark: since we are using tensor fields, we are free to perform every nonlinear change of coordinates. The convexity is hidden. For instance, in spherical coordinates we have:

$$\Lambda = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\otimes\frac{\partial}{\partial\varphi} \qquad \mathcal{R} = (1-r^2)\frac{\partial}{\partial r}\otimes\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\otimes\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\varphi}\otimes\frac{\partial}{\partial\varphi}$$

It is clear by inspection that Hamiltonian and gradient vector fields are tangent to the sphere of pure states, r=1.

The interior of the ball is an orbit of $SL(2,\mathbb{C})$

$\mathcal{S}$ is generated by $r\cos\theta\,,\ r\sin\theta\sin\varphi\,,\ r\sin\theta\cos\varphi$ by means of $\mathcal{R}\,,\ \Lambda$

Let us consider the Kossakowski-Lindblad equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \mathbf{L}(\rho)\,, \quad \rho(t=0) = \rho_0$$

$$\mathbf{L}(\rho) = -\imath\,[\mathbf{H}\,,\rho] + \frac{1}{2}\sum_j \left(\left[\mathbf{v}_j\rho\,,\mathbf{v}_j^\dagger\right] + \left[\mathbf{v}_j\,,\rho\mathbf{v}_j^\dagger\right]\right) =$$

$$= -\imath\,[\mathbf{H}\,,\rho] - \frac{1}{2}\sum_j \left[\mathbf{v}_j^\dagger\mathbf{v}_j\,,\rho\right]_+ + \sum_j \mathbf{v}_j\rho\mathbf{v}_j^\dagger$$

say with $\quad Tr(\mathbf{v}_j) = 0\,, \;\; Tr(\mathbf{v}_j^\dagger\mathbf{v}_k) = 0 \;\text{ if }\; j \neq k$

We see immediately that the equations of motion split into:

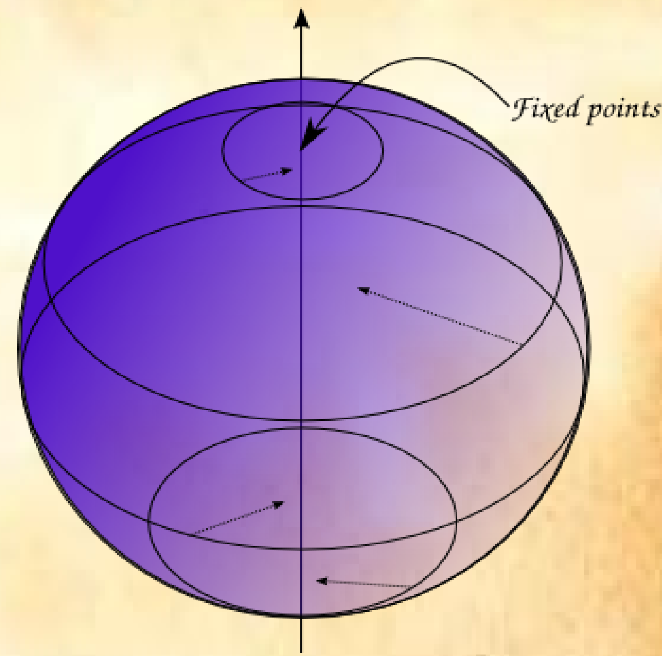
- *Hamiltonian term;*
- *Symmetric term;*
- *Kraus term.*

It is possible to write a vector field with this equation of motion. It turns out that the one associated with the Kraus term is a nonlinear vector field, similar to the nonlinear vector field associated with the symmetric tensor (the gradient vector field).

Example: the phase-damping of a q-bit

$$\mathbf{L}(\rho) = -\gamma\,(\sigma_3\,\rho\,\sigma_3 - \rho)$$

$$Z_L = -2\gamma\left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}\right)$$



The "miracle" of the Kossakowski-Lindblad equation is that the two nonlinearities camcel each other so that the resulting vector field is actually linear!

Summarizing

- *On the space of quantum states, Hamiltonian and gradient vector fields generate the action of a Lie group:* $SL(\mathcal{H}\,\mathbb{C})$
- *To describe semigroups we have to introduce Kraus vector fields.*
- *Having described the dynamics in terms of vector fields will provide a framework to describe non-Markovian dynamics.*
- *The tensorial description allows for generic nonlinear transformations, hopefully more flexible to deal with nonlinearities, like entanglement, entropies and so on.*