On the structure of projection-based tropical bases

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(Extended Abstract)

In the last years, tropical geometry has received much interest as a field combining aspects of algebraic geometry, discrete geometry, and computer algebra (for general background see, e.g., [5, 8, 9]). Given a field K with a real valuation $\text{ord} : K \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (i.e. $K = \mathbb{Q}$ with the p-adic valuation or the field $K = \mathbb{C}\{\{t\}\}$ of Puiseux series with the natural valuation) the valuation map extends to an algebraic closure $\bar{K}$ and to $\bar{K}^n$ via

$$\text{ord} : \bar{K}^n \to \bar{\mathbb{R}}^n, \quad (a_1, \ldots, a_n) \mapsto (\text{ord}(a_1), \ldots, \text{ord}(a_n)) .$$

Then for any polynomial $f = \sum_\alpha c_\alpha x^\alpha \in K[x_1, \ldots, x_n]$ the *tropicalization of* f is defined as

$$\text{trop}(f) = \bigoplus_\alpha \text{ord}(c_\alpha) \odot x^\alpha = \min_\alpha \{\text{ord}(c_\alpha) + \alpha_1 x_1 + \cdots + \alpha_n x_n\}$$

and the *tropical hypersurface* of f is

$$\mathcal{T}(f) = \{w \in \mathbb{R}^n : \text{the minimum in trop}(f) \text{ is attained at least twice in } w\} .$$

For an ideal $I \lhd K[x_1, \ldots, x_n]$, the *tropical variety of* I is given by

$$\mathcal{T}(I) = \bigcap_{f \in I} \mathcal{T}(f)$$

or equivalently (if the valuation is nontrivial) by the topological closure $\mathcal{T}(I) = \overline{\text{ord}\, \mathcal{V}(I)}$ where $\mathcal{V}(I) \subset (\bar{K}^*)^n$ is the variety of I.
From the viewpoint of computer algebra the natural way to handle an ideal is by means of a basis, i.e., a finite set of generators. In the stronger notion of a tropical basis it is additionally required that the set-theoretic intersection of

the tropical hypersurfaces of the generators coincides with the tropical variety. That is, a *tropical basis* of the ideal I is a finite generating set $\mathcal{F}$ of I, such that

$$\mathcal{T}(I) \;=\; \bigcap_{f \in \mathcal{F}} \mathcal{T}(f)\,.$$

The systematic study of tropical bases has been initiated by [1], [10]. These papers concentrate on the "constant coefficient case" (i.e., $I \lhd \mathbb{C}[x_1, \ldots, x_n]$) and provide Gröbner-related techniques. As a lower bound, it is shown in [1] that for $1 \leq d \leq n$ there is a d-dimensional linear ideal I in $\mathbb{C}[x_1, \ldots, x_n]$ such that any tropical basis of linear forms in I has size at least $\frac{1}{n-d+1}\binom{n}{d}$.
By revisiting the regular projection technique of Bieri and Groves [2], Hept and Theobald showed that every ideal has a short tropical basis [7].

Theorem 1 *Let $I \lhd K[x_1, \ldots, x_n]$ be a prime ideal generated by the polynomials $f_1, \ldots, f_r$. Then there exist $g_0, \ldots, g_{n-\dim I} \in I$ with*

$$\mathcal{T}(I) \;=\; \bigcap_{i=0}^{n-\dim I} \mathcal{T}(g_i)$$

and thus $\mathcal{G} := \{f_1, \ldots, f_r, g_0, \ldots, g_{n-\dim I}\}$ is a tropical basis for I of cardinality $r + \operatorname{codim} I + 1$.

This theorem can be seen as a tropical analogue to the Eisenbud-Evans-Theorem from classical algebraic geometry, which states that every algebraic set in n-space is the intersection of n hypersurfaces [4].

In this talk we present the main concept of these bases and report about ongoing work on understanding their structure.

Given an ideal $I \lhd K[x_1, \ldots, x_n]$ generated by polynomials $f_1, \ldots, f_r$, every tropical hypersurface in the proof of Theorem 1 has the form $\pi^{-1}\pi(\mathcal{T}(I))$ for some (rational) projection

$$\pi : \mathbb{R}^n \to \mathbb{R}^{\dim(I)+1}, \quad x \mapsto Ax$$

with a regular rational matrix A. The preimage $\pi^{-1}\pi(\mathcal{T}(I))$ is a tropical hypersurface which can be obtained in the following way. Let $u^{(1)}, \ldots, u^{(n-(m+1))} \in \mathbb{Q}^n$ be an (integer-valued) basis of the kernel of π. Denoting by J the ideal,

$$J := \Big\langle g \in K[x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_l] \;:$$

$$g = f(x_1 \prod_{j=1}^{n-(m+1)} \lambda_j{}^{u_1^{(j)}}, \ldots, x_n \prod_{j=1}^{n-(m+1} \lambda_j{}^{u_n^{(j)}}) \text{ for some } f \in I \Big\rangle\,,$$

we have

$$\pi^{-1}(\pi(\mathcal{T}(I))) \;=\; \mathcal{T}(J \cap K[x_1, \ldots, x_n])\,. \tag{1}$$

So $\mathcal{T}(J \cap K[x_1, \ldots, x_n])$ is a tropical hypersurface if the projection is m-dimensional. Hence, $\mathcal{T}(J \cap K[x_1, \ldots, x_n])$ naturally comes with a dual subdivision, and a first question is to characterize that subdivision (without computing the elimination ideal).
In order to study the subdivision of (1), we can apply recent results of Sturmfels, Yu, and Tevelev on tropical elimination and its relations to mixed fiber polytopes [11, 12]. In these papers, it is shown that in various situations the Newton polytope of the polynomial generating this hypersurface is affinely isomorphic to a mixed fiber polytope.
Here, we extend and refine this global viewpoint on the Newton polytope by studying as well the subdivision of the Newton polytope corresponding to the dual subdivision of (1). To establish this characterization, we provide some useful techniques to handle the affine isomorphisms connected with the mixed fiber polytopes.
In order to illustrate these ideas, let us denote the projection onto the kernel of π by ψ. Up to an affine isomorphism, the Newton polytope of $\pi^{-1}\pi(\mathcal{T}(I))$ is given by the mixed fiber polytope $\Sigma_\pi(\mathrm{New}(f_1), \ldots, \mathrm{New}(f_k))$. If all Newton polytopes coincide with some fixed Newton polytope Δ then the mixed fiber polytope can be written as

$$k!\ \Sigma_\psi(\Delta) \ = \ k! \int_{\psi(C)} (\psi^{-1}(x) \cap \Delta) dx \,.$$

However, computing the mixed fiber polytopes for the cells yields different affine isomorphims. Locally, in some simple cases the translation of the mixed fiber polytope is characterized by the following geometric result.

Theorem 2 *Let F be an $(n-1)$-polytope in $\mathbb{R}^n$, $\psi : \mathbb{R}^n \to \mathbb{R}$, $v \in \mathbb{R}^n \setminus \mathrm{aff}\, F$ and $P = \mathrm{conv}\{F \cup \{v\}\}$. Let w be an outer normal vector of the face F of P, i.e. $\mathrm{face}_w(P) = F$.*

- *(a) If $\psi(v) \in \psi(F)$ then $\Sigma_\psi(F)$ is a face of $\Sigma_\psi(P)$.*
- *(b) If $\psi(v) > \max_{x \in F} \psi(x)$ then*

$$\Sigma_\psi(F) + \sum_{\max_{x\in F} \psi(x)}^{\psi(v)-1} \arg \max_{x \in P \cap \psi^{-1}(i+\frac{1}{2})} w^T x \ = \ \mathrm{face}_w(\Sigma_\psi(P))\,. \qquad (2)$$

- *(c) If $\psi(v) < \min_{x \in F} \psi(x)$ then*

$$\Sigma_\psi(F) + \sum_{\psi(v)}^{\min_{x\in F} \psi(x)-1} \arg \max_{x \in P \cap \psi^{-1}(i+\frac{1}{2})} w^T x \ = \ \mathrm{face}_w(\Sigma_\psi(P))\,. \qquad (3)$$

Based on this result we can successively construct the subdivision of the mixed fiber polytope which is the dual subdivision of a tropical hypersurface of the tropical basis.

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