IDENTITIES OF BERNOULLI NUMBERS AND POLYNOMIALS

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In the field of Laurent series $\mathbb{Q}((T))$, the series $\mathbf{B} = T/(e^T - 1)$ is contained in the formal power series ring $\mathbb{Q}[[T]]$. The i-th Bernoulli number $B_i^{(n)}$ of order n is defined by

$$\mathbf{B}^n = \sum_{i=0}^{\infty} \frac{B_i^{(n)}}{i!} T^i.$$

We write $B_i^{(1)}$ simply as B_i. The i-th Bernoulli polynomial $B_i^{(n)}(X)$ of order n is defined by

$$\mathbf{B}^n e^{XT} = \sum_{i=0}^{\infty} \frac{B_i^{(n)}(X)}{i!} T^i \in \mathbb{Q}[X][[T]].$$

We are interested in identities involving Bernoulli numbers and polynomials, often of order one, such as

$$\sum_{i=1}^{n-1} (2n)! \frac{B_{2i}}{(2i)!} \frac{B_{2n-2i}}{(2n-2i)!} = -(2n+1)B_{2n} \qquad (n \geq 2)$$

discovered by Euler. These identities are better understood through Bernoulli numbers and polynomials of higher orders. In this talk, algorithms will be provided to produce identities including the following types.

Identity 1. *For $n \geq 3$,*

$$\sum_{\substack{i,l,k>0 \\ i+j+k=n}} (2n)! \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} \frac{B_{2c}}{(2c)!} = (n+1)(2n+1)B_{2n} + \frac{1}{2} n(2n-1)B_{2n-2}.$$

This generalization of Euler's identity is given by R. Sitaramachandrarao and B. Davis [8]. We obtain also other generalizations on sum of products of more Bernoulli numbers by K. Dilcher [2], A. Sankarayanan [7], R. Sitaramachandrarao and B. Davis [8], and W.-P. Zhang [9]. See also I-C. Huang and S.-Y. Huang [5].

Identity 2. *For $n \geq 4$*

$$\sum_{i=2}^{n-2} \frac{(2n-2)!}{(2i-2)!(2n-2i-2)!} \frac{B_{2i}}{2i} \frac{B_{2n-2i}}{2n-2i} = -\frac{(2n+1)(n-3)}{6n} B_{2n}.$$

This identity was proved by H. Rademacher [6] using Eisenstein series and by M. Eie [3] using Zeta functions. See also I-C. Huang and S.-Y. Huang [5].

Identity 3. *For $n \geq 2$,*

$$\begin{aligned} &\sum_{k=1}^{n-1} \binom{2n}{2k} 4^{2k} 6^{2n-2k} B_{2k} B_{2n-2k} \\ &= (2^{2n} - 4^{2n} - 6^{2n} - 2^{2n+1}n)B_{2n} + (16n)6^{2n-2} B_{2n-1}(\frac{1}{3}). \end{aligned}$$

This identity was discovered M. Eie [3]. See also I-C. Huang and S.-Y. Huang [5]. We are able to compute the sum

$$\sum_{\substack{i_1,i_2,\cdots,i_s\geq 0\\ i_1+i_2+\cdots+i_s=n}} \binom{n}{i_1,i_2,\cdots,i_s} m_1^{i_1}m_2^{i_2}\cdots m_s^{i_s}B_{i_1}B_{i_2}\cdots B_{i_s}$$

for given $m_1,m_2,\cdots,m_s$ and n. See [4] for the case $m_1=2$, $m_2=3$, $m_3=5$ and $m_4=6$.

Identity 4.

$$\sum_{i=0}^{n}\binom{n}{i}B_{2+i}B_{3+n-i}=\frac{1}{60}nB_{n+1}+\frac{1}{6}B_{n+3}-\frac{1}{60}(n+6)B_{n+5},$$

This is a special case of the identities considered in T. Agoh and K. Dilcher [1], which evaluates sums of the form $\sum_{i=0}^{n}\binom{n}{i}B_{m_1+i}B_{m_2+n-i}$. Our method applies to these sums as well as to

$$\sum_{\substack{i_1,i_2,\cdots,i_s\geq 0\\ i_1+i_2+\cdots+i_s=n}} \binom{n}{i_1,\cdots,i_s} B_{m_1+i_1}\cdots B_{m_s+i_s}.$$

References

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