Some new Techniques for Explicit Mixed Volume Computation

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Abstract

We give some new technical tools which simplify mixed volume computation for larger polynomial systems and allow the computation of mixed volume bounds for polynomial systems of arbitrary dimension arising in various applications as seen in [14].

1 Introduction

Let $f_1 = f_2 = \ldots = f_n = 0$ be a system of n polynomial equations in n variables. We are interested in the set of common solutions to this system in the case of a complete intersection. We denote by $S_i \subset \mathbb{Z}^n$ the support of f_i, i.e.

$$f_i(x_1, \ldots, x_n) = \sum_{\alpha \in S_i} c_\alpha x^\alpha$$

with $c_\alpha \neq 0 \in \mathbb{C}$ and where $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

In the following we want to use the support sets and purely combinatorial constructions on them to provide important algebraic informations.

Usually the systems we are interested in come from concrete applications and therefore have certain shape corresponding to the problem they model. It is desirable to calculate the mixed volume of these systems in dependence of the dimension of the problem. As an example the techniques of this paper are applied to an embedding problem of Laman graphs in [14]. The aim of this work is to present the new techniques from an abstract viewpoint without a certain application in the background.

The main tool we use is Bernstein's theorem which will be presented in section 3. In section 2 we will give a short introduction to mixed volumes and mixed subdivisions which will be used in section 5 to provide new results to simplify the computation of mixed volumes. In section 4 we present an efficient algorithm to compute the mixed volume.

2 Mixed Volumes and Mixed Subdivisions

We will give here a short introduction to the most important definitions and properties. More detailed descriptions can be found in [4], [8] or [13].

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Let $P_1, \ldots, P_n$ be n polytopes in $\mathbb{R}^n$. For non-negative parameters $\lambda_1, \ldots, \lambda_n$, the volume of the scaled Minkowski sum $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_n P_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$ with non-negative coefficients (see [16]). The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ is called the *mixed volume of* $P_1, \ldots, P_n$ and is denoted by $\mathrm{MV}_n(P_1, \ldots, P_n)$.

The mixed volume is linear in each argument, i.e.

$$\begin{aligned} &\mathrm{MV}_n(P_1, \ldots, \alpha P_i + \beta P_i', \ldots, P_n) = \\ &\qquad \alpha \, \mathrm{MV}_n(P_1, \ldots, P_i, \ldots, P_n) + \beta \, \mathrm{MV}_n(P_1, \ldots, P_i', \ldots, P_n) \end{aligned} \tag{1}$$

and it generalizes the usual volume in the sense that

$$\mathrm{MV}_n(P, \ldots, P) = \mathrm{vol}_n(P) \tag{2}$$

holds (see [13]).

We state here two explicit formulas for the mixed volume (see [15] and [8]):

$$\begin{aligned} &\mathrm{MV}_n(P_1, \ldots, P_n) \\ &\quad = (-1)^n \sum_{(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n} (-1)^{\sum_i \alpha_i} \, \mathrm{vol}_n \left(\sum_i \alpha_i P_i \right) && (3) \\ &\quad = \sum_{\substack{Q \text{ mixed cell of a} \\ \text{mixed subdivision} \\ \text{of } (P_1, \ldots, P_n)}} \mathrm{vol}_n(Q) && (4) \end{aligned}$$

The first formula (3) is obtained by using inclusion and exclusion formulas to compute the coefficient of $\lambda_1 \cdots \lambda_n$ in $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_n P_n)$, see [13]. To understand the second formula (4) and for further considerations we have to introduce the reader to mixed subdivisions. For technical reasons we prefer here to define mixed subdivisions on point sets rather then on polytopes. This definition can then easily be extended to polytopes by considering their vertex sets as point sets.

Let $\mathcal{S} = (S^{(1)}, \ldots, S^{(m)})$ be a sequence of finite point sets in $\mathbb{R}^n$ that affinely spans the full space. A sequence $C = (C^{(1)}, \ldots, C^{(m)})$ of subsets $C^{(i)} \subseteq S^{(i)}$ is called a *cell* of $\mathcal{S}$. A *subdivision* of $\mathcal{S}$ is a collection $\Gamma = (C_1, \ldots, C_k)$ of cells such that

i) $\dim(\mathrm{conv}(C_i)) = n$ for all cells C_i,

ii) $\mathrm{conv}(C_i) \cap \mathrm{conv}(C_j)$ is a face of both convex hulls and

iii) $\bigcup_{i=1}^k \mathrm{conv}(C_i) = \mathrm{conv}(\mathcal{S})$

where $\mathrm{conv}(A) := \mathrm{conv}(A^{(1)} + \ldots + A^{(m)})$ for a sequence of point sets A. A subdivision is called *mixed* if additionally

iv) $\sum_{i=1}^m \dim(\mathrm{conv}(C_j^{(i)})) = n$ for all cells C_j in Γ

and it is called *fine mixed* if additionally

v) $\sum_{i=1}^m (|C_j^{(i)}| - 1)) = n$ for all cells C_j in Γ

where $|A|$ denotes the number of points in a finite set $A \subset \mathbb{R}^n$. The *type* of a cell is defined as

$$type(C) = (\dim(\operatorname{conv}(C^{(1)})), \ldots, \dim(\operatorname{conv}(C^{(m)})))$$

and cells of type $(1, 1, \ldots, 1)$ will be called *mixed cells*. These definitions extend naturally to sequences of polytopes by considering their vertices as the point sets above. In this case every mixed subdivision will be a fine mixed subdivision. If all cells of a subdivision are simplices we will call the subdivision a *triangulation*.

To construct mixed subdivisions we proceed as in [8]. Not every subdivision can be constructed in this way but since we will only need one arbitrary mixed subdivision we will use this simple construction. For each of the point sets $S^{(i)}$ from $\mathcal{S}$ we choose a linear lifting function $\mu_i : \mathbb{R}^n \to \mathbb{R}$ identified by an element of $\mathbb{R}^n$. By $\hat{A}$ we denote the lifted point sets $\{(q, \langle \mu_i, q \rangle) : q \in A\} \in \mathbb{R}^{n+1}$.

The set of those facets of $\operatorname{conv}(\hat{S}^{(1)} + \ldots + \hat{S}^{(m)})$ which have an inward pointing normal with a positive last coordinate is called the lower hull of the Minkowski sum. If we project down this lower hull back to $\mathbb{R}^n$ by forgetting the last coordinate we get a subdivision of $(S^{(1)}, \ldots, S^{(m)})$. We call such a subdivision *coherent* and will say it is *induced by* $\mu_1, \ldots, \mu_m$.

For the subdivision induced by $\mu_1, \ldots, \mu_m$ to be a fine mixed subdivision it is sufficient that every vertex of the lower envelope can be expressed uniquely as a Minkowski sum (see [4]). Such a set of liftings will be called *(sufficiently) generic*.

3 Bernstein's Theorem

The core theorem that gives a connection between solutions to systems of polynomial equations and discrete geometry is the following.

Theorem 1 (Bernstein [1]) *Given polynomials $f_1, \ldots, f_n$ over $\mathbb{C}$ with finitely many common zeroes in $(\mathbb{C}^*)^n$, let P_i denote the Newton polytope (i.e. the convex hull of the support set) of f_i in $\mathbb{R}^n$. Then the number of common zeroes of the f_i in $(\mathbb{C}^*)^n$ is bounded above by the mixed volume $MV_n(P_1, \ldots, P_n)$. Moreover for generic choices of the coefficients in the f_i, the number of common solutions is exactly $MV_n(P_1, \ldots, P_n)$.*

Bernstein also gives an explicit condition when a choice of coefficients is generic. A refinement of these conditions is due to Canny and Rojas (see [2]). It should be mentioned that the theorem also holds for Laurent polynomials.

Various attempts have been made to generalize these results to count all common roots in $\mathbb{C}^n$ (see for example [5], [9] and [11]). The easiest, but sometimes not the best bound is $\mathrm{MV}_n(\operatorname{conv}(P_1 \cup 0), \ldots, \operatorname{conv}(P_n \cup 0))$ which is shown in [11].

The bound on the number of solutions of a polynomial system arising from Bernstein's theorem is also often referred to as the *BKK bound* due to the work of Bernstein, Khovanskii and Kushnirenko. The BKK bound generalizes the Bezout bound (see [3] chapter 7) and for sparse polynomial systems it is often significantly better. As an example consider the following eigenvalue problem. We want to solve $Av = \lambda v$ where $A \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ of unit length.

This gives rise to a system of $n+1$ quadratic equations in $n+1$ variables.

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} v_j - \lambda v_i &= 0 \quad \text{for all } i = 1, \ldots, n \\ \sum_{i=1}^{n} v_i^2 &= 1 \end{aligned}$$

Bezout's theorem says that this system has at most 2^{n+1} solutions while the mixed volume of the corresponding Newton polytopes can be computed to be $2n$ which is even a sharp bound since each eigenspace cuts the unit sphere in 2 points.

4 An Algorithm for Computing the Mixed Volume

In this section we want to give a sketch of a state of the art algorithm to compute the mixed volume. A detailed description can be found in [4].

We assume that we already have a generic lifting μ_i for each polytope P_i ($i = 1, \ldots, n$) in the sense of section 2. Again, the lifted polytopes will be denoted by $\hat{P}_i$ and the Minkowski sum of the P_i is denoted by P.

The idea for the computation of $\mathrm{MV}(P_1, \ldots, P_n)$ is then the following. For each combination of n edges e_i from the given polytopes we test, whether their lifted Minkowski sum lies on the lower envelope of $\hat{P}$. If so, we compute the volume of the corresponding mixed cell and add it to the mixed volume. To make this naive algorithm efficient we will make use of the following simple Lemma (see [4]).

Lemma 2 *If $\sum_{j \in J} \hat{e}_j$ lies on the lower envelope of $\sum_{j \in J} \hat{P}_j$ then $\sum_{t \in T} \hat{e}_t$ lies on the lower envelope of $\sum_{t \in T} \hat{P}_t$ for every subset $T \subset J$.*

So instead of performing a few expensive tests on the sum of n edges we do many small tests to build up valid sums of edges step by step. Each test for a k-tuple of edges $e_1, \ldots, e_k$ will be implemented as a linear program as follows. Let $\hat{m}_i \in \mathbb{R}^{n+1}$ denote the midpoint of the lifted edge $\hat{e}_i$ of $\hat{P}_i$ such that $\hat{m} = \hat{m}_1 + \ldots + \hat{m}_k$ is an interior point of the Minkowski sum $\hat{e}_1 + \ldots + \hat{e}_k$. Consider the following linear program.

$$\begin{aligned} \text{maximize } s &\in \mathbb{R}_{\geq 0} \\ \text{s.t. } \hat{m} &- (0, \ldots, 0, s) \in \hat{P}_1 + \ldots + \hat{P}_k \end{aligned} \tag{5}$$

If we denote the vertices of P_i by $v_1^{(i)}, \ldots, v_{r_i}^{(i)}$ this can be written as

$$\begin{aligned} \text{maximize} \quad & s \in \mathbb{R}_{\geq 0} \\ \text{s.t.} \quad & \hat{m} - (0, \ldots, 0, s) = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \lambda_j^{(i)} \hat{v}_j^{(i)} \\ & \sum_{j=1}^{r_i} \lambda_j^{(i)} = 1 \quad \forall\, i = 1, ..., n \\ & \lambda_j^{(i)} \geq 0 \quad \forall\, i, j \quad . \end{aligned}$$

s measures the distance of $\hat{m}$ to the lower envelope of the Minkowski sum. Hence $\hat{m}$ lies on the lower envelope of $\hat{P}_1 + \ldots + \hat{P}_k$ if and only if the optimal value of (5) is zero.

The worst case complexity of the algorithm arising from these ideas is in $r^{O(n)}$ where r denotes the maximal number of vertices of the P_i. In general it can be shown that mixed volume computation is #P-complete [12]. Since computing the volume of the convex hull of a point set is #P-hard (see [10]) and since the mixed volume is a generalization of the volume (see (2)) this is the best we could hope for.

5 Technical Tools to Simplify the Computation

Polynomial systems that arise from inductive structures often have a shape that allows to separate the mixed volume computation in several parts. The following Lemma is most often used in the special case when all polytopes involved have integer vertices. In this case there is a much shorter proof using Bernstein's theorem. However we would like to state it here in the general case and give a purely geometric proof for it.

Lemma 3 *Let $P_1, \ldots, P_k$ be polytopes in $\mathbb{R}^{m+k}$ and $Q_1, \ldots, Q_m$ be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then*

$$\mathrm{MV}_{m+k}(Q_1, ..., Q_m, P_1, ..., P_k) = \mathrm{MV}_m(Q_1, ..., Q_m) * \mathrm{MV}_k(\pi(P_1), ..., \pi(P_k)) \tag{6}$$

where $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ denotes the projection on the last k coordinates.

Proof. Lemma 4.5 in [6] shows that

$$\mathrm{MV}_{m+k}(Q, \ldots, Q, P, \ldots, P) = \mathrm{vol}_m(Q) * \mathrm{vol}_k(\pi(P)) \tag{7}$$

where Q is taken m times and P is taken k times. So this proves the case where $Q_1 = \ldots = Q_m = Q$ and $P_1 = \ldots = P_k = P$.

Now we will show that both sides of the desired equation define a symmetric multilinear function and then we will use combinatorial identities for symmetric multilinear functions to show the full result.

Let $\mathcal{P}^m$ (resp. $\mathcal{P}^{m+k}$) be the set of all m-dimensional (resp. $m+k$-dimensional) polytopes and define two functions g_1 and g_2 on $\mathcal{P}^m \times \ldots \times \mathcal{P}^m \times \mathcal{P}^{m+k} \times \ldots \times \mathcal{P}^{m+k}$ via

$$\begin{aligned} g_1(Q_1, \ldots, Q_m, P_1, \ldots, P_k) &:= \mathrm{MV}_{m+k}(Q_1, \ldots, Q_m, P_1, \ldots, P_k) \\ g_2(Q_1, \ldots, Q_m, P_1, \ldots, P_k) &:= \mathrm{MV}_m(Q_1, ..., Q_m) * \mathrm{MV}_k(\pi(P_1), ..., \pi(P_k)) \ . \end{aligned}$$

It is easy to see that g_1 and g_2 are invariant under changing the order of the Q_i and also changing the order of the P_j. Furthermore it follows from (1) that both functions are linear in each argument. Let $f : A \times \ldots \times A \to B$ be a symmetric multilinear function, where A and B are semigroups. By expanding the right hand side it can be seen that

$$f(a_1, ..., a_n) = \frac{1}{n!} \sum_{1 \le i_1 < \ldots < i_q \le n} (-1)^{n-q} f(a_{i_1} + ... + a_{i_q}, ..., a_{i_1} + ... + a_{i_q}) \ . \tag{8}$$

The functions

$$\begin{aligned} \tilde{g}_i(Q_1,\dots,Q_m) &:= g_i(Q_1,\dots,Q_m,P_1,\dots,P_k) \text{ and} \\ \bar{g}_i^{(Q)}(P_1,\dots,P_k) &:= g_i(Q,\dots,Q,P_1,\dots,P_k) \text{ for } i=1,2 \end{aligned}$$

satisfy these conditions. Hence we have for $i=1,2$ that

$$\begin{aligned} & g_i(Q_1,\dots,Q_m,P_1,\dots,P_k) \\ &= \tilde{g}_i(Q_1,\dots,Q_m) \\ &= \frac{1}{m!} \sum_{1\le i_1<\dots<i_q\le m} (-1)^{m-q}\ \tilde{g}_i(Q_{i_1}+\dots+Q_{i_q},\dots,Q_{i_1}+\dots+Q_{i_q}) \\ &= \frac{1}{m!} \sum_{1\le i_1<\dots<i_q\le m} (-1)^{m-q}\ \bar{g}_i^{(Q_{i_1}+\dots+Q_{i_q})}(P_1,\dots,P_k)\ . \end{aligned}$$

Since we can expand $\bar{g}_i^{(Q_{i_1}+\dots+Q_{i_q})}(P_1,\dots,P_k)$ by using (8) as well we see that both functions g_1 and g_2 are fully determined by their images of tuples of polytopes where $Q_1=\dots=Q_m=Q$ and $P_1=\dots=P_k=P$. This proves the Lemma. □

Since the mixed volume does not change if all arguments are mapped by the same volume preserving function (see [3]) we have the following corollary.

Corollary 4 *Let $P_1,\dots,P_n$ be polytopes in $\mathbb{R}^n$ such that the first m of them lie in an m-dimensional subspace V of $\mathbb{R}^n$. Then*

$$\mathrm{MV}_n(P_1,...,P_n) = \mathrm{MV}_m(\pi_V(P_1),...,\pi_V(P_m)) * \mathrm{MV}_{n-m}(\pi_{V^*}(P_{m+1}),...,\pi_{V^*}(P_n))$$

where π_V and π_{V^*} denote the projection to V and to the orthogonal complement V^* of V respectively.

Another technical tool which can be useful is the following Lemma which gives explicit conditions for lifting vectors to induce a certain cell as a mixed cell.

Lemma 5 *Given polytopes $P_1,\dots,P_n\subset\mathbb{R}^n$ and lifting vectors $\mu_1,\dots,\mu_n\in\mathbb{R}^n_{\ge 0}$. Denote the vertices of P_i by $v_1^{(i)},\dots,v_{r_i}^{(i)}$ and choose one edge $e_i=[v_{k_i}^{(i)},v_{l_i}^{(i)}]$ from each P_i. Then $C:=(e_1,\dots,e_n)$ is a mixed cell of the mixed subdivision induced by the liftings μ_i if and only if*

i) The edge matrix $E:=V_a-V_b$ is non-singular (where $V_a:=(v_{k_1}^{(1)},\dots,v_{k_n}^{(n)})$ and $V_b:=(v_{l_1}^{(1)},\dots,v_{l_n}^{(n)})$) and

ii) For all polytopes P_i and all vertices $v_s^{(i)}$ of P_i which are not in e_i we have:

$$\left(\mathrm{diag}\left(\mu^T E\right)^T E^{-1} - \mu_i^T\right)\cdot\left(v_{l_i}^{(i)} - v_s^{(i)}\right) \ge 0 \qquad (9)$$

where $\mu:=(\mu_1,\dots,\mu_n)$ and where $\mathrm{diag}(V)$ denotes the vector of the diagonal entries of V.

Before we start with the proof we have to give a short introduction to linear programming and how it is applied here. (For details on linear programming see [7].) In section 4 we have seen that the test, if a cell lies on the lower envelope of the lifted Minkowski sum can be formulated as a linear program (see (5)). We will speak of a linear program in standard form if it is stated as follows

$$\begin{array}{rl} \text{maximize} & c^t.x \\ \text{s.t.} & A.x = b \\ & x_i \geq 0 \quad \forall i = 1, \dots, n \end{array}$$

where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{n \times m}$. Given a feasible point $\bar{x} \geq 0$ satisfying $A.\bar{x} = b$ we want to check whether $\bar{x}$ is an optimal solution. If $\bar{x}$ is a vertex of the polyhedron defined by the constraints and is not degenerate in the sense defined below, we can use linear programming duality to test for optimality. To $\bar{x}$ corresponds a choice B (sometimes more than one) of columns of A such that the submatrix consisting of these columns A_B satisfies $A_B^{-1}.b = \bar{x}$. ($\bar{x}$ is non-degenerate if the inverse of A_B exists.) Let A_N be the submatrix of A consisting of the remaining columns and define c_B and c_N in the same way. Then $\bar{x}$ is a feasible point of the dual program and therefore optimal if and only if

$$c_N^t - c_B^t.A_B^{-1}.A_N \leq 0 \quad \text{(componentwise)} . \tag{10}$$

Our linear program (5) can be written in standard form using the following notation.

$$\begin{array}{rcl} c^t & = & (\mathbf{0}^t_{r_1+\ldots+r_n}, 1) \; \in \mathbb{R}^{r_1+\ldots+r_n+1} \\ x^t & = & (\lambda_1^{(1)}, \dots, \lambda_{r_1}^{(1)}, \dots\dots, \lambda_1^{(n)}, \dots, \lambda_{r_n}^{(n)}, s) \; \in \mathbb{R}^{r_1+\ldots+r_n+1} \\ b^t & = & (\hat{m}, \mathbf{1}_n^t) \; \in \mathbb{R}^{2n+1} \\ A & = & \begin{pmatrix} v_1^{(1)} & \dots & v_{r_1}^{(1)} & \dots & \dots & v_1^{(n)} & \dots & v_{r_n}^{(n)} & \mathbf{0}_n \\ \langle \mu_1, v_1^{(1)} \rangle & \dots & \langle \mu_1, v_{r_1}^{(1)} \rangle & \dots & \dots & \langle \mu_n, v_1^{(n)} \rangle & \dots & \langle \mu_n, v_{r_n}^{(n)} \rangle & 1 \\ & \mathbf{1}^t_{r_1} & & \mathbf{0}^t_{r_2} & \dots & & \mathbf{0}^t_{r_n} & & 0 \\ & \mathbf{0}^t_{r_1} & & \mathbf{1}^t_{r_2} & \dots & & \mathbf{0}^t_{r_n} & & 0 \\ & \vdots & & & \ddots & & \vdots & & \vdots \\ & \mathbf{0}^t_{r_1} & & \mathbf{0}^t_{r_2} & \dots & & \mathbf{1}^t_{r_n} & & 0 \end{pmatrix} \end{array}$$

Here we denote by $\mathbf{0}_n$ and $\mathbf{1}_n$ the column vectors consisting only of 0's and 1's respectively.

We also know that the point $\bar{x}$ that corresponds to $\hat{m}$ in (5) must be a vertex of the feasible region since all matrix inequalities are satisfied with equality. The coordinates $(\lambda_1^{(1)}, \dots\dots, \lambda_{r_n}^{(n)}, s)$ of this point are $s = 0$ and $\lambda_j^{(i)} = \frac{1}{2}$ if the edge $\hat{e}_i$ contains the vertex $\hat{v}_j^{(i)}$ and $\lambda_j^{(i)} = 0$ otherwise. If the edges were chosen such that their Minkowski sum is full-dimensional we will see later that we can also guarantee that this $\bar{x}$ is non-degenerate. We can use now the condition (10) on this description to obtain explicit conditions on the lifting vectors μ_i that induce a mixed subdivision that contains our chosen cell as a mixed cell.

Proof. (of Lemma 5) Note that C is full-dimensional and hence has a non-zero volume if and only if E is non-singular. In the following we will only consider this case. To simplify the notation we write $\mu(V)$ for $\operatorname{diag}(\mu^t.V)^t$.

We know from section 4 that C is a mixed cell if and only if the following $\bar{x}$ is the optimal solution to the linear program defined above.

$$\bar{x} = (\lambda_{1,1}, \ldots, \lambda_{n,r_n}, s)$$

where $s = 0$ and

$$\lambda_{i,j} = \begin{cases} \frac{1}{2} & , j = k_i \text{ or } l_i \\ 0 & , \text{else} \end{cases}$$

The submatrix of A corresponding to $\bar{x}$ is

$$A_B = \begin{pmatrix} V_a & V_b & \mathbf{0}_n \\ \mu(V_a) & \mu(V_b) & 1 \\ \mathrm{Id}_n & \mathrm{Id}_n & \mathbf{0}_n \end{pmatrix}$$

and its inverse is

$$A_B^{-1} = \begin{pmatrix} (E)^{-1} & \mathbf{0}_n & -(E)^{-1}.V_b \\ -(E)^{-1} & \mathbf{0}_n & (E)^{-1}.V_a \\ -\mu(E).(E)^{-1} & 1 & \mu(E).(E)^{-1}.V_b - \mu(V_b) \end{pmatrix} .$$

A_N consists of the columns of A which are not in A_B, hence

$$A_N = \begin{pmatrix} v_s^{(i)} \\ \mu_r.v_s^{(i)} \\ \xi_i \end{pmatrix}_{\substack{1 \leq i \leq n \\ 1 \leq s \leq r_i \\ s \neq k_i, l_i}}$$

where ξ_r denotes the i^{th} unit vector. Since $c_N = (0, \ldots, 0)$ the simplex criterion tells us that $\bar{x}$ is optimal if

$$(0, \ldots, 0, 1).A_B^{-1}.A_N \geq 0 \quad \text{componentwise} .$$

But a single entry of the vector on the left can be explicitly computed as

$$-\left(\mu(E).(E)^{-1}\right).v_s^{(i)} + \mu_r.v_s^{(i)} + \left(\mu(E).(E)^{-1}.V_b - \mu(V_b)\right).\xi_i$$

which equals

$$\left(\mathrm{diag}\left(\mu^t.(E)\right)^t.(E)^{-1} - \mu_i^t\right).\left(v_{l_i}^{(i)} - v_s^{(i)}\right) .$$

□

Note that (9) is linear in the μ_j. Hence given a choice of edges we can explicitly calculate $\sum_{i=1}^n r_i$ normal vectors defining a cone in $\mathbb{R}^{n^2}$. The interior of this cone consists of all liftings $(\mu_1^t, \ldots, \mu_n^t)$ which induce a mixed subdivision that contains our chosen cell as a mixed cell.

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