SPECIAL ORBIFOLDS AND BIRATIONAL CLASSIFICATION. LEVICO. GGA VIII

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- **Decomposition problem :** "split" X canonically into its "pure" parts by fibrations.

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- Conjecturally, special orbifolds enjoy the same expected properties as manifolds with $\kappa_+ = -\infty$ or $\kappa = 0$. (Because of their expected stability under "orbifold extension").

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- Geometric orbifolds generalise this construction.

GEOMETRIC ORBIFOLDS. OBJECTS

- **Geometric orbifolds=** pairs (X/Δ), $\Delta = \sum_J a_j.D_j$, J finite, $(1 - 1/m_j) = a_j \in]0, 1] \cap \mathbb{Q}, D_j$ irreducible divisors, $m_j \in]1, +\infty]$:=multiplicities. (Same objects as in LMMP, but from (apparently) different motivations).

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- $K_{(X/\Delta)} := K_X + \Delta$, $\kappa(X/\Delta) := \kappa(X, K_X + \Delta)$.

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- Let $\kappa(f) := \kappa(Y'/\Delta_{f'})$, for $f' : X' \to Y'$ obtained by flattening and smoothing of f. (Not needed if $\kappa(Y) \geq 0$)

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- Multiple fibres not always eliminated by étale covers.

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  3. $\kappa(X) = 1$, and $q(X') \leq 1, \forall X'$, étale cover of X

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- **Orbifold Kobayashi-Ochiai :**

  $[\exists \varphi : \mathbb{C}^n \dashrightarrow X$ meromorphic non-degenerate$] \Longrightarrow X$ special.

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- **(A-CT vs C Undecided)**. But hyperbolic analogue of **C** known for some B-T's.

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- Either **A-CT**, or the expected link arithmetics-hyperbolicity fails.

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- **2nd step :** For a general choice of D : the number of rational or elliptic curves m-tangent to D is finite. (Since $\kappa(S/\Delta_f) = 2$ with $(c_1^2 - c_2)(S/\Delta_f) > 0)$: orbifold Bogomolov).

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- **If** (X/Δ) **integral,** one can define also : $\pi_1(X/\Delta)$, $d_{(X/\Delta)}$, integral points if (X/Δ) defined over a number field.

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- $f : (X/\Delta) \dashrightarrow Y$ **of general type** if $\kappa(f/\Delta) = dim(Y) > 0$.

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- **Theorem :** (Orbifold variant of Bogomolov (1978))
  $L \subset \Omega^p_X, p > 0$ rank one coherent. Then :
  1. $\kappa(X/\Delta, L) \leq p$. ($L$:=**Bogomolov** iff equality).
  2. $\kappa(X/\Delta, L) = p \Longleftrightarrow \exists f : X \dashrightarrow Y$ s.t : $L = f^*(K_Y)$ generically over Y, with : $\kappa(f/\Delta) = dim(Y) = p$).

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- Let $f : X \to Y$ be a fibration, and $L_f := f^*(K_Y)$. Then : $\kappa(X/\Delta, L_f) = \kappa(f/\Delta)$. (On **any** model).
- **Theorem :** (Orbifold variant of Bogomolov (1978))
  $L \subset \Omega^p_X, p > 0$ rank one coherent. Then :
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- $f \to L_f$ induces natural bijection :
  [f of Δ-**general type**]$\to$ Δ-**Bogomolov-sheaves.**

BOGOMOLOV SHEAVES

- Let (X/Δ) smooth, $L \subset \Omega^p_X$ rank one, coherent, L^m the **saturation** in $S^N(\Omega^p_{(X/\Delta)})$ of $L^{\otimes m}$, for $m > 0$.
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- Needs **non-classical** multiplicities (a major motivation).

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- $\mathbf{C}_{n,m}^{orb}$ **Conjecture :** $f : (X/\Delta) \to Y$ a fibration, (X/Δ) smooth. Then : $\kappa(X/\Delta) \geq \kappa(X_y/\Delta_y) + \kappa(f/\Delta)$?

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- Special $\Longleftrightarrow$ (canonical) tower of fibrations with orbifold fibres having either $\kappa = 0$, or $\kappa_+ = -\infty$. (Conditionally in $C_{n,m}^{orb}$)

- **Theorem :** Let (X/Δ) smooth. There exists a unique fibration $c = c_{(X/\Delta)} : (X/\Delta) \to C = C(X/\Delta)$ s.t :
  1. Its general orbifold fibres are special.
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- Conjecturally, **c** splits arithmetics and hyperbolicity as well.

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- **Corollary :** Assume $C^{orb}_{n,m}$.
  Let $\mathcal{P}$ be a class of smooth orbifolds which :
  1. is birationally stable.
  2. contains all orbifolds with either $\kappa_+ = -\infty$ or $\kappa = 0$.
  3. is stable by extensions.

  • Then : $\mathcal{P} \supset \mathcal{S}$ (the class of special orbifolds).

  • And $\mathcal{P} = \mathcal{S}$ if, moreover :

  4. $\mathcal{P}$ does not contain any orbifold of general type.
  5. $\mathcal{P}$ is stable by image.

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- In these last two cases, only 1. is known.
  For 3., the local obstructions vanish.
  Expectations : global obstructions do not exist.
  And 2. is the orbifold extension of standard conjectures.

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- X a **smooth** surface, $f : X \to \mathbb{P}_1$ a fibration : Δ_f^* divides Δ_f. $(\Delta_f^* \neq \Delta_f$ and $\kappa(f) = 1) \Longrightarrow \kappa(X) = 2$.

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- *abc* $\Longrightarrow$ Orbifold Mordell $\Longrightarrow$ yes. $(f(X(k)) \subset (\mathbb{P}_1/\Delta_f)(k))$

ORBIFOLD MORDELL AND abc

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  $\{x = a/b : (a, b, c) \in \mathbb{Z}, (a, b) = 1, a + b = c$, s.t :
  a is p-full, b is r-full, and c is q-full$\}$, where :
  a **p-full** means : m prime divides a, then : m^p divides a.
  $Card\{a \leq N, a = p - full\} \sim_{N \to +\infty} C_p.N^{1/p}$.

- Let $(\mathbb{P}_1/\Delta)$, let $p, q, r > 1$ be integers, and :
  $\Delta := (1 - 1/p).\{0\} + (1 - 1/q).\{1\} + (1 - 1/r).\{\infty\}$.
  $\kappa(X/\Delta) = 1 \iff (1/p + 1/q + 1/r) < 1$. (Over $k = \mathbb{Q}$).
- **Classical** $\mathbb{Q}$-rational points : $(X/\Delta)^*(\mathbb{Q}) :=$
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ORBIFOLD MORDELL AND abc

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- $abc \implies$ Orbifold Mordell (Open. True over function fields)