# Families of rational curves and classification of Fano manifolds

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# DEFINITIONS



$X$ smooth complex projective variety of dimension $n$

$$X \text{ Fano manifold} \iff -K_X \text{ ample.}$$

- $r_X$ index of $X$

$$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL\}$$

- $i_X$, pseudoindex of $X$

$$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m\}.$$

$r_X$ and $i_X$ positive integers $\leq n+1$

# Definitions



$X$ Fano variety of index $r_X$

$$\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$$

and it is torsion free. Its rank $\rho$ is called Picard number of $X$.

There is a unique line bundle $L \in \mathrm{Pic}(X)$ such that

$$-K_X = r_X L$$

$L$ is called fundamental divisor of $X$.

# Classification: low dimensions

Fano manifolds of fixed dimension form a bounded family.

The classification is known in low dimensions

**Curves and surfaces**

| | |
|---|---|
| 1 | $\mathbb{P}^1$ |
| 2 | del Pezzo surfaces |

**Threefolds**

| | |
|---|---|
| $\rho = 1$ | Fano, Iskovskikh |
| $\rho \geq 2$ | Mori & Mukai |

The classification in case $\rho = 1$ is based on two facts:

- $|L|$ contains a smooth $S$ (del Pezzo or $K3$)
- $X$ contains a line, i.e. a curve $C$ such that $-K_X \cdot C = r_X$

The classification in case $\rho \geq 2$ is obtained via Mori theory, using that

- Either $X$ is a blow up of a Fano $X'$ or has a conic bundle structure over a smooth surface.

# Classification: high index



## Theorem (Kobayashi & Ochiai)

*$X$ Fano manifold of index $r_X$. Then $r_X \leq \dim X + 1$ and*

- $r_X = \dim X + 1$ *if and only if* $X \simeq \mathbb{P}^n$;
- $r_X = \dim X$ *if and only if* $X \simeq \mathbb{Q}^n$.

The classification of Fano manifolds is also known in the two subsequent cases:

## High index

$r_X = \dim X - 1$ del Pezzo manifolds;
$r_X = \dim X - 2$ Mukai manifolds.

# APOLLONIUS METHOD



$X$ Fano variety of index $\dim X - k$

$$-K_X = (\dim X - k)L$$

$L$ fundamental divisor of $X$

If there exists a smooth $X' \in |L|$ (a good divisor) then

$$-K_{X'} = (-K_X - L)'_X = (\dim X' - k)L_{X'}$$

$X'$ is a Fano manifold of the same coindex.

del Pezzo and Mukai manifolds have good divisors, so there is a ladder going down to surfaces and threefolds, respectively.

# Fano manifolds of index $\dim X - 3$



## $\rho = 1$

| | |
|---|---|
| Good divisors ($\dim X = 4$) | $\|L\|$ contains a (singular) Calabi-Yau |
| Good divisors ($\dim X > 4$) | Unknown |
| Lines | Unknown; |

## $\rho \geq 2$

| | |
|---|---|
| $\dim X = 4$ | Unknown |

# FANO MANIFOLDS OF INDEX $\dim X - 3$



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### $\rho \geq 2$

| | |
|---|---|
| $\dim X = 4$ | Unknown |
| $\dim X \geq 9$ | None |

# Fano manifolds of index $\dim X - 3$



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**$\rho \geq 2$**

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|---|---|
| $\dim X = 4$ | Unknown |
| $\dim X = 8$ | $\mathbb{P}^4 \times \mathbb{P}^4$ |
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**Theorem (Wisniewski)**

If $r_X \geq (n+2)/2$ then $\rho \leq 2$, equality iff $X \simeq \mathbb{P}^{n/2} \times \mathbb{P}^{n/2}$

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A classification of the border cases is known

## Middle index and $\rho_X \geq 2$

| | |
|---|---|
| $r_X = (n+1)/2$ | Wiśniewski |
| $r_X = n/2$ | Wiśniewski, Ballico, Peternell, Szurek |

# FANO MANIFOLDS OF INDEX $\dim X - 3$



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## $\rho \geq 2$

| | |
|---|---|
| $\dim X = 4$ | Unknown |
| $\dim X = 6, 7$ | Classified |
| $\dim X = 8$ | $\mathbb{P}^4 \times \mathbb{P}^4$ |
| $\dim X \geq 9$ | None |

## THEOREM (WISNIEWSKI)

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### Idea

- Except for $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ these varieties have $\rho = 2$;
- Study and compare the two extremal contractions.

# FANO MANIFOLDS OF INDEX $\dim X - 3$



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## $\rho \geq 2$

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|---|---|
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| $\dim X = 5$ | ... |
| $\dim X = 6, 7$ | Classified |
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## IDEA

- Except for $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ these varieties have $\rho = 2$;
- Study and compare the two extremal contractions.

## Aim

Classify Fano fivefolds of index $\geq 2$ and Picard number $\geq 2$

Joint works with Andreatta, Chierici, Novelli (almost any possible subset)

## Strategy

1. Give a bound on $\rho$;
2. Classify the possible cones of curves;
3. Classify the varieties.

# Families of rational curves



$\mathrm{Hom}(\mathbb{P}^1, X)$ scheme parametrizing $f : \mathbb{P}^1 \to X$

$\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$ open subset

$\mathrm{Ratcurves}^n(X)$ quotient of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ by $\mathrm{Aut}(\mathbb{P}^1)$

Family of rational curves: $V \subset \mathrm{Ratcurves}^n(X)$ irreducible

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "\pi"'] & X \\ V & \end{tikzcd}$$

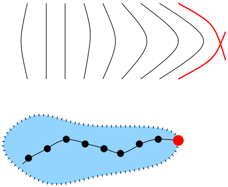
$$\mathrm{Locus}(V) = i(U), \; V_x = \pi(i^{-1}(x))$$

- $V$ unsplit if $V$ is proper;
- $V$ locally unsplit if $V_x$ is proper for a general $x$ in $\mathrm{Locus}(V)$.

# Chow families



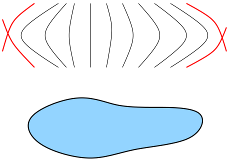
$$\mathrm{Ratcurves}^n(X) \to \mathrm{Chow}(X)$$



$$V \rightsquigarrow \overline{V} = \mathcal{V} \subset \mathrm{Chow}(X)$$

Reducible cycles are parametrized by points in $\mathcal{V} \setminus V$

Chow family of rational curves: $\mathcal{V} \subset \mathrm{Chow}(X)$ irreducible, parametrizing rational and connected 1-cycles.



If $V$ is an unsplit family by abuse $V = \mathcal{V}$.

# CHAINS OF RATIONAL CURVES



$Y \subset X$ closed, $\mathscr{V}^1, \ldots, \mathscr{V}^k$ Chow families

## DEFINITION

$\mathrm{Locus}(\mathscr{V}^1, \ldots, \mathscr{V}^k)_Y$: points $x \in X$ s.t. there exists $C_1, \ldots, C_k$

- $C_i$ belongs to $\mathscr{V}^i$
- $C_i \cap C_{i+1} \neq \emptyset$
- $C_1 \cap Y \neq \emptyset$ and $x \in C_k$

## DEFINITION

$\mathrm{ChLocus}_m(\mathscr{V}^1, \ldots, \mathscr{V}^k)_Y$: points $x \in X$ such that there exists $C_1, \ldots, C_m$

- $C_i$ belongs to $\mathscr{V}^j$
- $C_i \cap C_{i+1} \neq \emptyset$
- $C_1 \cap Y \neq \emptyset$ e $x \in C_m$

# GOOD PROPERTIES OF CHAINS - I



$X$ smooth $\mathscr{V}$ unsplit family

- $\dim \operatorname{Locus}(\mathscr{V}) + \dim \operatorname{Locus}(\mathscr{V}_x) \geq \dim X - K_X \cdot \mathscr{V} - 1$;
- $\dim \operatorname{Locus}(\mathscr{V}_x) \geq -K_X \cdot \mathscr{V} - 1$.

$Y \subset X$ closed $\mathscr{V}$ unsplit family numerically independent from curves in $Y$

- $\dim \operatorname{Locus}(\mathscr{V})_Y \geq \dim Y - K_X \cdot \mathscr{V} - 1$ (if $\neq \emptyset$)

$\mathscr{V}^1, \ldots, \mathscr{V}^k$ numerically independent unsplit families, with $< [\mathscr{V}^1], \ldots, [\mathscr{V}^k] >$ independent from curves in $Y$

- $\dim \operatorname{Locus}(\mathscr{V}^1, \ldots, \mathscr{V}^k)_Y \geq \dim Y - \sum K_X \cdot \mathscr{V}^i - k$ (if $\neq \emptyset$)

# Good properties of chains - II



### Lemma (Numerical equivalence)

*$Y \subset X$ closed, $\mathcal{V}^1, \dots, \mathcal{V}^k$ Chow families, $C$ curve contained in $\mathrm{ChLocus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$.*

$$C \equiv aC_Y + \sum b_j C_{\mathcal{V}^j}$$

*$a, b \in \mathbb{Q}$, $C_Y \subset Y$ and $C_{\mathcal{V}^j}$ irreducible component of a cycle in $\mathcal{V}^j$.*

### Lemma (Numerical equivalence improved)

*$Y \subset X$ closed and "extremal", $\mathcal{V}$ unsplit family, $C \subset \mathrm{Locus}(\mathcal{V})_Y$ curve.*

$$C \equiv aC_Y + bC_{\mathcal{V}} \qquad a, b \in \mathbb{Q}_{\geq 0}$$

# RC fibrations



$\mathscr{V}^1, \dots, \mathscr{V}^k$ Chow families

$x$ and $y$ are $\mathrm{rc}(\mathscr{V}^1, \dots, \mathscr{V}^k)$ equivalent if either $x = y$ or there is a chain of curves in $\mathscr{V}^1, \dots, \mathscr{V}^k$ joining $x$ and $y$, i.e. for some $m$

$$y \in \mathrm{ChLocus}_m(\mathscr{V}^1, \dots, \mathscr{V}^k)_x.$$

**Theorem (Campana, Kollár-Miyaoka-Mori)**

*There exists $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to Z^0$ such that*

- *Fibers of $\pi$ are equivalence classes*
- *$\forall z \in Z^0$ two points in $\pi^{-1}(z)$ are connected by at most $2^{\dim X - \dim Z} - 1$ cycles in $\mathscr{V}^1, \dots, \mathscr{V}^k$*

**Corollary**

*$X$ $\mathrm{rc}(\mathscr{V}^1, \dots, \mathscr{V}^k)$ connected; every curve in $X$ is equivalent to a combination of classes of components of cycles in $\mathscr{V}^1, \dots, \mathscr{V}^k$.*
*If $\mathscr{V}^1, \dots, \mathscr{V}^k$ are unsplit then $\rho \le k$.*

# Quasi unsplit families

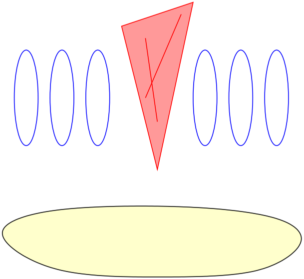
We don't need $\mathscr{V}$ unsplit.

A Chow family $\mathscr{V}$ is called quasi unsplit if the irreducible components of cicles in $\mathscr{V}$ are numerically proportional

In $\mathbb{P}^2 \times \mathbb{P}^3$

$$X = \{x_0{}^2 y_0 + x_1{}^2 y_1 + x_2{}^2 y_2 = 0\},$$

$\mathscr{V}$ family of conics given by the intersection of $X$ with fibers of the first projection



# RATIONAL CURVES ON FANO MANIFOLDS



### THEOREM (MORI)

*$X$ Fano; $\forall x \in X$ there is a rational curve $C$ through $x$ with $-K_X \cdot C \leq \dim X + 1$.*

### THEOREM (KOLLÁR-MIYAOKA-MORI)

*$X$ Fano, $\pi : X^0 \to Z^0$ proper surjective morphism; for a general $z \in Z^0$ there is a rational curve $C$ with $-K_X \cdot C \leq \dim X + 1$ s.t.*

- *$C \cap \pi^{-1}(z) \neq \emptyset$*
- *$C$ is not contained in $\pi^{-1}(z)$*

$V^i \subset \mathrm{Ratcurves}^n(X)$ of anticanonical degree $\leq \dim X + 1$ are a finite number $\Rightarrow$ there exists $i$ s.t. $\mathrm{Locus}(V^i)$ dominates $X$ (resp. $Z^0$). A family of minimal degree with this property is called a minimal dominating family (resp a minimal horizontal dominating family) and it is locally unsplit.

# Bounding the Picard number



## Strategy

1. Give a bound on $\rho$;
2. Classify the possible cones of curves;
3. Classify the varieties.

# Mukai conjecture



**Mukai**

$$\rho_X(r_X - 1) \le \dim X$$

**Generalized Mukai - (BCDD)**

$$\rho_X(i_X - 1) \le \dim X,$$

equality iff $X \simeq (\mathbb{P}^{i_X - 1})^{\rho_X}$

**Theorem**

*$X$ Fano of dimension five. Then*

$$\rho_X(i_X - 1) \le 5$$

*equality holding iff $X \simeq (\mathbb{P}^1)^5$*

# Ideal case

## There's always a quasi unsplit family when you need one



Let $V^1$ be a minimal dominating family for $X$.

Suppose that $\mathcal{V}^1$ is quasi unsplit and consider the rc$\mathcal{V}^1$ fibration $\pi^1 : X^1 \to Z^1$.

If $\dim Z^1 = 0$ then $\rho = 1$, else

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Suppose that $\mathcal{V}^2$ is quasi unsplit and consider the rc$(\mathcal{V}^1, \mathcal{V}^2)$ fibration $\pi^2 : X^2 \to Z^2$.

If $\dim Z^2 = 0$ then $\rho = 2$, else

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If $\dim Z^2 = 0$ then $\rho = 2$, else

let $V^3$ be a minimal horizontal dominating family for $\pi^2$

...

$$\dim \operatorname{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_x \geq -\sum K_X \cdot \mathcal{V}^i - k \geq k$$

so we finish in at most five steps.

# A TASTE OF REAL WORLD



Let $V^1$ be a minimal dominating family for $X$.

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Suppose that $\mathscr{V}^2$ is not quasi unsplit.

In $\mathscr{V}^2$ there is a reducible cycle $C_1 + C_2$ with $[C_1] \neq \lambda[V^2]$.

$$-K_X \cdot V^2 = -K_X \cdot (C_1 + C_2) \geq 2i_X \geq 4$$

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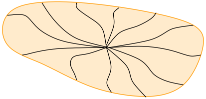
$$-K_X \cdot V^2 = -K_X \cdot (C_1 + C_2) \geq 2 i_X \geq 4$$

pick a general $x \in \text{Locus}(V^2)$ and let $Y = \text{Locus}(V^2)_x$

- $\dim Y \geq -K_X \cdot V^2 - 1 \geq 3$
- Every curve in $Y$ is proportional to $[V^2]$



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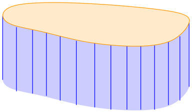

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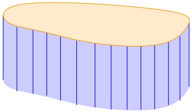
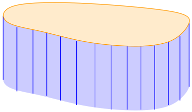


If $\mathrm{Locus}(\mathscr{V}^1)_Y = X$ then $\rho = 2$, so $D$ has dimension four.

# A TASTE OF REAL WORLD





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Let $\pi^2 : X^2 \to Z^2$ be the $\text{rc}(\mathscr{V}^1, \mathscr{V}^2)$ fibration.

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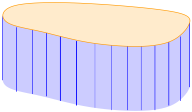

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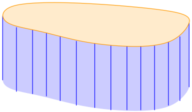

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$$\dim \mathrm{Locus}(V_3)_z = 1$$

which implies

- $-K_X \cdot V^3 = 2$
- $\dim \mathrm{Locus}(V^3) = 5$

against the minimality of $V^2$.

# A taste of real world



So $X$ is $\mathrm{rc}(\mathscr{V}^1, \mathscr{V}^2)$ connected and every curve in $X$ is numerically proportional to a combination of components of cycles in $\mathscr{V}^1$ and $\mathscr{V}^2$.

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$$\dim \mathrm{Locus}(W^i)_x \geq \dim X - \dim \mathrm{Locus}(W^i) + i_X - 1 \geq 2$$

any $\Gamma \subset \mathrm{Locus}(W^i)_x$ is proportional to $W^i \Rightarrow D \cap \mathrm{Locus}(W^i)_x = \emptyset$.

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$D \cdot V^2 = 0$ $D \cdot V^2 = D \cdot (C_1 + C_2) = 0.$

Conclusion: $D \equiv 0$, a contradiction.

# Description of the Mori cone



## Strategy

1. Give a bound on $\rho$;
2. Classify the possible cones of curves;
3. Classify the varieties.

# Description of the Mori cone

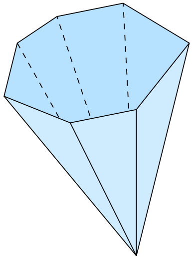


The cone of curves of a Fano manifold $X$ is closed and polyhedral, spanned by a finite number of rays, in the vector space $N_1(X) \simeq \mathbb{R}^\rho$.

Every ray corresponds to a contraction, i.e. to a morphism with connected fiber onto a normal variety $W$ such that the relative Picard number is one.

Kinds of contractions

- Fiber type contractions
- Divisorial contractions
- Small contractions



Description of the cone: find the number and type of the extremal rays.

# An (easy) example



Suppose that $\rho_X = 3$, two rays $R_1$ and $R_2$ are of fiber type and that the contraction associated to another ray, $R_3$ has a three dimensional fiber $G$.

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$$\boxed{\mathrm{NE}(X) = \langle R_1, R_2, R_3 \rangle}$$

# Classification of the cones



| $\rho_X$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ |
|---|---|---|---|---|---|
| 2 | $F$ | $F$ | | | |
| | $F$ | $D_0$ | | | |
| | $F$ | $D_1$ | | | |
| | $F$ | $D_2$ | | | |
| | $F$ | $S$ | | | |
| | $D_2$ | $D_2$ | | | |
| | $D_2$ | $S$ | | | |
| 3 | $F$ | $F$ | $F$ | | |
| | $F$ | $F$ | $S$ | | |
| | $F$ | $F$ | $D_1$ | | |
| | $F$ | $F$ | $D_2$ | | |
| | $F$ | $D_2$ | $D_2$ | | |
| 4 | $F$ | $F$ | $F$ | $F$ | |
| | $F$ | $F$ | $F$ | $D_2$ | |
| 5 | $F$ | $F$ | $F$ | $F$ | $F$ |

$F$ *fiber type*

$D_i$ *divisor to $i$-dim subvariety*

$S$ *small*

# EFFECTIVE CLASSIFICATION



## STRATEGY

1. Give a bound on $\rho$;
2. Classify the possible cones of curves;
3. Classify the varieties.

# Example
## Fivefolds without fiber type contractions



| $\rho_X$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ |
|---|---|---|---|---|---|
| 2 | $\dots$<br>$D_2$<br>$D_2$ | $\dots$<br>$D_2$<br>$S$ | | | |

1. Prove that the $D_2$ contractions are smooth blow-ups (easy);
2. Prove that $X$ is the blow up of $\mathbb{P}^5$ along a smooth surface;
   - Fix a minimal dominating family in the target;
   - Compare its pullback with the families on $X$;
   - Show that the family has degree 6.
3. Classify the possible surfaces.

# EXAMPLE
## FIVEFOLDS WITHOUT FIBER TYPE CONTRACTIONS



| $\rho_X$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ |
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# Example

## Fivefolds without fiber type contractions



$S \subset \mathbb{P}^5$ center of the blow-up, $\Gamma$ $k$-secant line of $S$, $\widetilde{\Gamma}$ proper transform of $\Gamma$; then

$$-K_X \cdot \widetilde{\Gamma} = -K_{\mathbb{P}^5} \cdot \Gamma - 2E \cdot \widetilde{\Gamma} = 6 - 2k$$

- $S$ has no trisecants;
- $Sec(S) \neq \mathbb{P}^5$.

1. $S$ nondegenerate $\Longrightarrow$ $S$ Veronese.
2. $S \subset \mathbb{P}^4$ + $\widetilde{\mathbb{P}^4}$ is exceptional $\Longrightarrow$ $S$ cubic scroll
3. $S \subset \mathbb{P}^3 \Longrightarrow S \simeq \mathbb{Q}^2$

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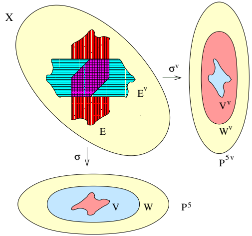


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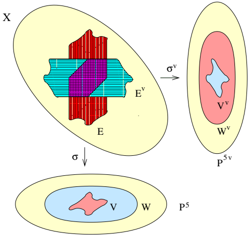


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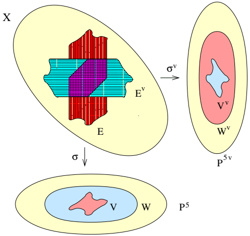


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# Classification so far



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|---|---|---|---|---|---|
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| | $F$ | $D_0$ | | | |
| | $F$ | $D_1$ | | | |
| | $F$ | $D_2$ | | | |
| | $F$ | $S$ | | | |
| | $D_2$ | $D_2$ | | | |
| | $D_2$ | $S$ | | | |
| 3 | $F$ | $F$ | $F$ | | |
| | $F$ | $F$ | $S$ | | |
| | $F$ | $F$ | $D_1$ | | |
| | $F$ | $F$ | $D_2$ | | |
| | $F$ | $D_2$ | $D_2$ | | |
| 4 | $F$ | $F$ | $F$ | $F$ | |
| | $F$ | $F$ | $F$ | $D_2$ | |
| 5 | $F$ | $F$ | $F$ | $F$ | $F$ |

# Classification so far



| | |
|---|---|
| $\langle D_2, S\rangle$ | Blow up of $\mathbb{P}^5$ along a two dimensional quadric |
| $\langle D_2, D_2\rangle$ | Blow up of $\mathbb{P}^5$ along a Veronese surface |
| | Blow up of $\mathbb{P}^5$ along a cubic scroll $\subset \mathbb{P}^4$ |
| $\langle F, F, F\rangle$ | A general member of $\mathscr{O}(1,1,1) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ |
| | The intersection of two general members of $\mathscr{O}(1,0,1)$ and $\mathscr{O}(0,1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ |
| | $\mathbb{P}^1 \times Y$ (No. 1-9, 11, 14) |
| $\langle F, F, S\rangle$ | $Bl_p(\mathbb{P}^4) \times_{\mathbb{P}^3} Bl_p(\mathbb{P}^4)$ |
| $\langle F, F, D_1\rangle$ | $\mathbb{P}^1 \times Y$ (No. 13, 15) |
| $\langle F, F, D_2\rangle$ | Blow up of a general member of $\mathscr{O}(1,1) \subset \mathbb{P}^2 \times \mathbb{P}^4$ along a section of the first projection |
| | $\mathbb{P}^1 \times Y$ (No. 10, 12) |
| $\langle F, D_2, D_2\rangle$ | $Bl_S(Bl_p(\mathbb{P}^5))$ with $S$ the strict trasform of a $\mathbb{P}^2 \ni p$ |
| | Blow up of a cone in $\mathbb{P}^9$ over the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ along its vertex |
| | Blow up of $\mathbb{P}^5$ in two non meeting 2-planes |
| $\langle F, F, F, F\rangle$ | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2)$ |
| $\langle F, F, F, D_2\rangle$ | $\mathbb{P}^1 \times \mathbb{P}^1 \times Bl_p(\mathbb{P}^3)$ |
| $\langle F, F, F, F, F\rangle$ | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ |