Mukai conjecture for Fano manifolds and related problems

Gianluca Occhetta

Dipartimento di Matematica - Università di Trento

KIAS Workshop
Rational Curves on Algebraic Varieties

Numerical invariants

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r_X and i_X positive integers $\leq \dim X + 1$

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FAMILIES OF RATIONAL CURVES

Wiśniewski's proof - Ingredients

Families of rational curves

$$\mathrm{Hom}(\mathbb{P}^1, X) \quad \text{scheme parametrizing } f : \mathbb{P}^1 \to X$$
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- V locally unsplit if V_x is proper for a general x in $\mathrm{Locus}(V)$.

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A family of minimal degree with this property is called a minimal dominating family and it is locally unsplit.

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- There exists a face of $\mathrm{NE}(X)$ containing two extremal rays with meeting exceptional loci.

Chow families

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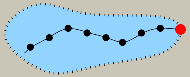

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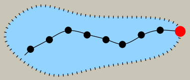

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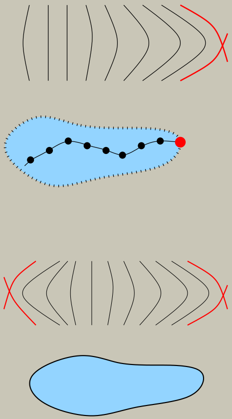

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- C_i belongs to $\mathscr{V}^j$
- $C_i \cap C_{i+1} \neq \emptyset$
- $C_1 \cap x \neq \emptyset$ e $y \in C_m$

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Theorem (Campana, Kollár-Miyaoka-Mori)

There exists $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to Z^0$ such that fibers of π are equivalence classes and $\forall z \in Z^0$ two points in $\pi^{-1}(z)$ are connected by at most $2^{\dim X - \dim Z} - 1$ cycles in $\mathscr{V}^1, \ldots, \mathscr{V}^k$

NUMERICAL EQUIVALENCE - III

Special varieties - Ingredients

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$$C \equiv \sum b_j C_{\mathscr{V}^j}$$

$b_j \in \mathbb{Q}$ and $C_{\mathscr{V}^j}$ irreducible component of a cycle in $\mathscr{V}^j$.

Special varieties - Ingredients

Numerical equivalence - III

$\mathscr{V}^1, \ldots, \mathscr{V}^k$ Chow families, C curve contained in $\mathrm{ChLocus}(\mathscr{V}^1, \ldots, \mathscr{V}^k)_x$.

$$C \equiv \sum b_j C_{\mathscr{V}^j}$$

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X $rc(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ connected; every curve in X is equivalent to a combination of classes of components of cycles in $\mathscr{V}^1, \ldots, \mathscr{V}^k$.

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9. All the families are covering.

Special varieties I - Conclusion

We conclude by the following

Theorem

A smooth complex projective variety X of dimension n is isomorphic to $\mathbb{P}^{n(1)} \times \cdots \times \mathbb{P}^{n(k)}$

if and only if

$\exists V^1, \ldots, V^k$ unsplit and covering with $\sum -K_X \cdot V^k = n + k$ such that $\dim \langle [V^1], \ldots, [V^k] \rangle = k$ in $N_1(X)$.

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6. In fact, if $\Gamma \in V$ then $z \in \mathrm{Locus}(V)$.
7. V_z cannot be unsplit since $\dim \mathrm{Locus}(V_z) \cap \mathrm{Locus}(V^1, V^2)_y > 0$.

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but not the second, because there can exist reducible cycles in $\mathscr{V}$ whose components are proportional to $[V]$.

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$V^1, \ldots, V^k$ unsplit with $\dim < [V^1], \ldots, [V^k] >= k$ and $\mathrm{NE}(Y) \cap < [V^1], \ldots, [V^k] >= \underline{0}$

- $\dim \mathrm{Locus}(\mathscr{V}^1, \ldots, \mathscr{V}^k)_Y \geq \dim Y - \sum K_X \cdot \mathscr{V}^i - k \qquad (\text{if} \neq \emptyset)$

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Quasi unsplit families

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Fivefolds - Ingredients

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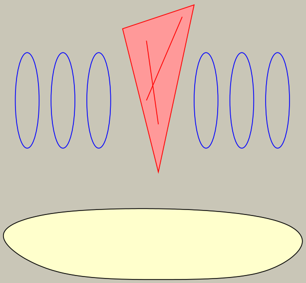

Fivefolds - A case of the proof

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10. X is $\mathrm{rc}(\mathscr{V}, \mathscr{V}')$ connected.

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1. W^i families of irreducible components of reducible cycles in $\mathscr{V}'$.

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4. Conclusion: $D \equiv 0$, a contradiction.

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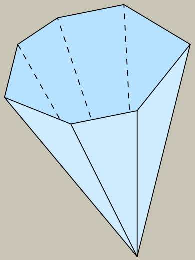

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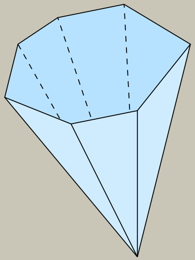

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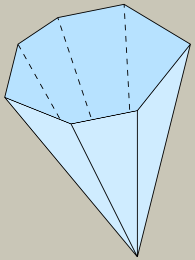

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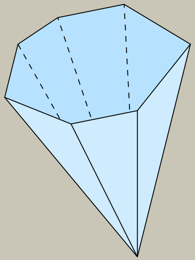

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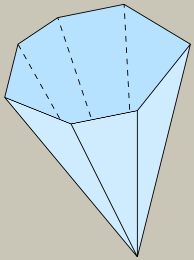

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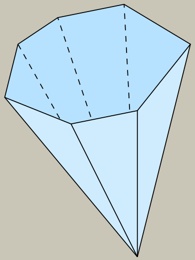

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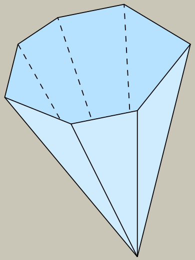

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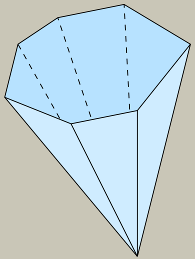
Aim: find the number and type of the extremal rays.

Description of the Mori cone

The cone of curves of a Fano manifold X is closed and polyhedral, spanned by a finite number of rays, in the vector space $N_1(X) \simeq \mathbb{R}^\rho$.



Every ray corresponds to a contraction, i.e. to a morphism with connected fiber onto a normal variety such that the relative Picard number is one.

Kinds of contractions

- Fiber type contractions
- Divisorial contractions
- Small contractions

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Skip example

An (easy) example

Suppose that $\rho_X = 3$, two rays R_1 and R_2 are of fiber type and that the contraction associated to another ray, R_3 has a three dimensional fiber G.

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Classification of the cones

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ρ_X	R_1	R_2	R_3	R_4	R_5
2	F	F			
	F	S			
	F	D_0			
	F	D_1			
	F	D_2			
	D_2	D_2			
	D_2	S			
3	F	F	F		
	F	F	S		
	F	F	D_1		
	F	F	D_2		
	F	D_2	D_2		
4	F	F	F	F	
	F	F	F	D_2	
5	F	F	F	F	F

F *fiber type*

D_i *divisor to i-dim subvariety*

S *small*

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Cones with complete classification

	F	D_2			
	D_2	D_2			
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	F	F	S		
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	F	D_2	D_2		
4	F	F	F	F	
	F	F	F	D_2	
5	F	F	F	F	F

Classification so far

Non trivial cases - I

$\langle D_2, S\rangle$	Blow up of $\mathbb{P}^5$ along a two dimensional quadric
$\langle D_2, D_2\rangle$	Blow up of $\mathbb{P}^5$ along a Veronese surface
	Blow up of $\mathbb{P}^5$ along a cubic scroll $\subset \mathbb{P}^4$
$\langle F, F, F\rangle$	A general member of $\mathscr{O}(1,1,1) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$
	The intersection of two general members of $\mathscr{O}(1,0,1)$ and $\mathscr{O}(0,1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$
$\langle F, F, S\rangle$	$Bl_{\mathrm{p}}(\mathbb{P}^4) \times_{\mathbb{P}^3} Bl_{\mathrm{p}}(\mathbb{P}^4)$
$\langle F, F, D_2\rangle$	Blow up of a general member of $\mathscr{O}(1,1) \subset \mathbb{P}^2 \times \mathbb{P}^4$ along a section of the first projection
$\langle F, D_2, D_2\rangle$	$Bl_{\mathrm{S}}(Bl_{\mathrm{p}}(\mathbb{P}^5))$ with S the strict trasform of a $\mathbb{P}^2 \ni p$
	Blow up of a cone in $\mathbb{P}^9$ over the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ along its vertex
	Blow up of $\mathbb{P}^5$ in two non meeting 2-planes

Classification so far

Non trivial cases - II

$\langle F, D_2 \rangle$	Blow up of $\mathbb{P}^5$ along a point
	Blow up of $\mathbb{P}^5$ along the complete intersection of three quadrics
	Blow up of $\mathbb{P}^5$ along $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}(1,2)$
	Blow up of $\mathbb{P}^5$ along $S \simeq \mathbb{F}_2$ embedded by $C_0 + 3f$
	Blow up of $\mathbb{P}^5$ along the blow-up of $\mathbb{P}^2$ in four points $x_1, \dots, x_4$ embedded by $\mathcal{O}_{\mathbb{P}}^2(3) - \sum x_i$
	Blow up of $\mathbb{P}^5$ along the blow-up of $\mathbb{P}^2$ in seven points $x_0, \dots, x_6$ embedded by $\mathcal{O}_{\mathbb{P}}^2(3) - 2x_0 - \sum_{i=1}^6 x_i$
	Blow up of a del Pezzo fivefold V_d of degree $d \leq 5$ along a del Pezzo surface of degree d
	Blow up of V^3 along a plane
	Blow up of V^4 along a plane
	Blow up of V^4 along a quadric
	Blow up of V^5 along a σ-plane
	Blow up of V^5 along a quadric

A bound on the length

Another related problem is the following: which is the maximal length of an extremal ray of a Fano manifold X of Picard number $\rho_X \geq 2$ and pseudoindex i_X?

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The proof is based on the following

Lemma

X Fano with $\rho_X \geq 2$, R extremal ray, $\mathrm{Exc}(R)$ exceptional locus. Then $\exists V$, $[V] \notin R$ such that $\mathrm{Exc}(R) \cap \mathrm{Locus}(V) \neq \emptyset$ and, for some $x \in \mathrm{Exc}(R) \cap \mathrm{Locus}(V)$, V_x is proper.

Skip proof

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9. Set $W^1 = T^k$.

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PROOF

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8. Else there exists a C_k independent from R which meets $\mathrm{Exc}(R)$.
9. Set $W^1 = T^k$.
10. If $W^1_{x_1}$ is unsplit we set $V = W^1$, else repeat the argument.

Fano manifolds with long rays - I

Theorem

If equality holds and R is of fiber type or divisorial then

- $X \simeq \mathbb{P}^k \times \mathbb{P}^{n-k}$
- $X \simeq Bl_{\mathbb{P}^t}(\mathbb{P}^n)$ with $0 \leq t \leq \frac{n-3}{2}$.

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Theorem

If $\quad r_X + l(R) = \dim \mathrm{Exc}(R) + 2 \quad$ then, if $e = \dim \mathrm{Exc}(R)$
$X = \mathbb{P}_{\mathbb{P}^k}(\mathscr{O}^{\oplus e-k+1} \oplus \mathscr{O}(1)^{\oplus n-e})$, with $k = n - r + 1$.

FANO MANIFOLDS WITH LONG RAYS - II

We also have some results in the case

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The best one is for blow-ups:

THEOREM

X Fano; $\varphi_R : X \to Y$ blow up of a smooth Y along a smooth $T \subset Y$, such that $i_X \geq \dim T + 1$. Then X is

- $Bl_{\mathbb{P}^t}(\mathbb{P}^n)$, with $\mathbb{P}^t$ a linear subspace of dimension $\leq \frac{n}{2} - 1$,
- $Bl_{\mathbb{P}^t}(\mathbb{Q}^n)$, with $\mathbb{P}^t$ a linear subspace of dimension $\leq \frac{n}{2} - 1$,
- $Bl_{\mathbb{Q}^t}(\mathbb{Q}^n)$, with $\mathbb{Q}^t$ a smooth quadric of dimension $\leq \frac{n}{2} - 1$ not contained in a linear subspace of $\mathbb{Q}^n$,
- $Bl_p(V)$ with $V \simeq Bl_Y(\mathbb{P}^n)$ and Y submanifold of dimension $n-2$ and deg $\leq n$ contained in an hyperplane H s.t. $p \notin H$,
- $Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$.