PROJECTIVE MANIFOLDS CONTAINING A LARGE LINEAR SUBSPACE WITH NEF NORMAL BUNDLE

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Birational automorphism groups and birational geometry,
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Setup

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Problem: classify X under suitable assumptions on s and $\mathscr{N}$.

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In case (2) one can apply a consequence of Zak's Theorem on Tangencies.

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In case (1) the result is obtained by studying the Hilbert scheme of s-planes in X.

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THEOREM (BELTRAMETTI - SOMMESE - WIŚNIEWSKI)

Assume that X is covered by a family of lines of anticanonical degree $\geq \frac{n+3}{2}$. Then there is an extremal ray of $\mathrm{NE}(X)$ generated by the numerical class of such a line.

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By results of Mori theory, if $\varphi : X \to Y$ is a fiber type contraction associated to a (negative) extremal ray R, then X is covered by rational curves whose numerical class is in R.

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By taking a minimal degree dominating family of rational curves in R one obtains a quasi-unsplit family.

Quasi-unsplit: curves in the family can degenerate to reducible cycles, but every irreducible component of such a cycle is numerically proportional to a curve in the family.

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1. X is toric;
2. The dimension of a general V-equivalence class is $\geq \dim X - 3$.

Recent results

Theorem (Novelli, _)

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The proof combines the main idea of B-S-W (studying the nefness of a suitable adjoint divisor) and the description of the indeterminacy locus of the rationally connected fibration associated to the family of lines given in B-C-D.

Corollary

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By adjunction

$$-K_X \cdot l = s+1+c,$$

so we can apply the theorem.

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2. a smooth hyperquadric;
3. If s is even $\mathbb{G}(1, s+1)$.

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$X^{2s} \subset \mathbb{P}^N$. There exists a linear s-space contained in X with nef normal bundle iff X is covered by linear spaces of dimension s.

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By standard arguments of Hilbert schemes it follows that X is covered by linear spaces of dimension s.

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Then he showed that

1. X is a projective bundle over a smooth variety;
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3. if s is even, then X is $\mathbb{G}(1,s+1)$.

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IDEA

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- Nice structure of the exceptional divisor;
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- $\widetilde{X}$ is a Fano manifold of Picard number 2;
- it is possible to study $\widetilde{X}$ by studying the "second contraction".

Blowing-up Grassmannians

Λ^s linear subspace of $\mathbb{G}(1,s+1)$ parametrizes the star of lines through a point P and its normal bundle is $T_\Lambda(-1)$.
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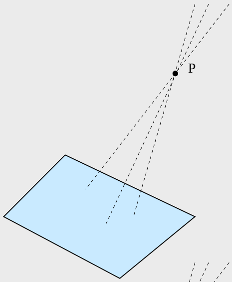

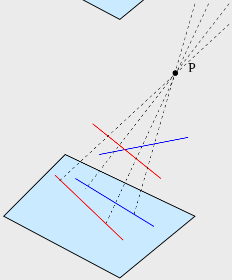

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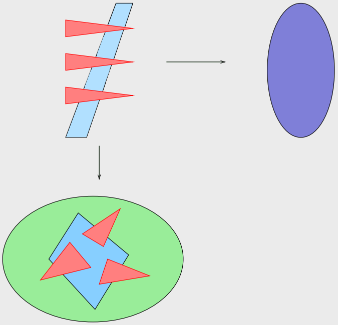

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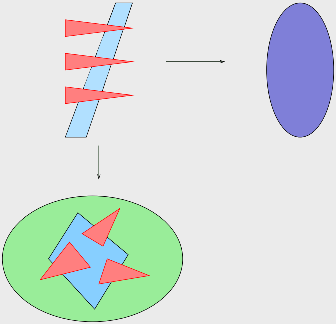

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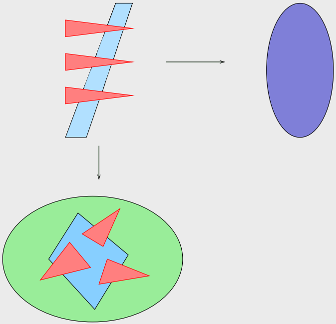

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It can be shown that $\widetilde{\mathbb{G}}(1,s+1) = \mathbb{P}_{\mathbb{G}(1,\mathscr{H})}(\mathscr{Q} \oplus \mathscr{O}(1))$, where $\mathscr{Q}$ is the universal quotient bundle over $\mathbb{G}(1,\mathscr{H})$.

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Picard number

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Contradiction (no linear spaces with normal $T(-1)$).

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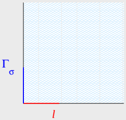

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Therefore the restriction of $\widetilde{\pi}$ to E is the $\mathbb{P}^1$-bundle $\varphi : E \to \mathbb{G}(1,s)$ coming from the incidence diagram.

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Moreover $2H - E$ is ample and $(2H-E)|_F \simeq \mathscr{O}_{\mathbb{P}^2}(1)$; therefore $\widetilde{\pi}$ is a $\mathbb{P}^2$-bundle over $\mathbb{G}(1,s)$ by a result of Fujita.

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By the canonical bundle formula for projective bundles

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Since $h^1(\mathscr{Q}^*(1)) = h^1(\mathscr{Q}) = 0$, the sequence splits and

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Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s+1$, containing a linear subspace Λ of dimension s, such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of X is one, then X is one of the following:

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$\mathscr{N}$ is uniform of type $(0,\dots,0,1,\dots,1)$, again by the sequence

$$0 \to \mathscr{N} \to N_{\Lambda/\mathbb{P}^N} = \mathscr{O}_\Lambda(1)^{\oplus(N-s)} \to (N_{X/\mathbb{P}^N})|_X \to 0$$

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$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\sigma"'] & Y \\ X \arrow[ur, dashed, "\pi"'] & \end{tikzcd}$$

$l \subset X$ line meeting Λ but not in it.
$\widetilde{\pi}$ contracts ℓ, the strict transform of l.

$$\mathrm{NE}(\widetilde{X}) = \langle [\Gamma_\sigma], [\ell] \rangle$$

and the supporting divisors of the rays are $H = \sigma^*\mathcal{O}(1)$ and $H - E$.

Big difference: $(H - E)_E$ is ample!

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Since $n \geq 4$ we have a contradiction.

Last case - Sketch(es) of proof

The remaining case

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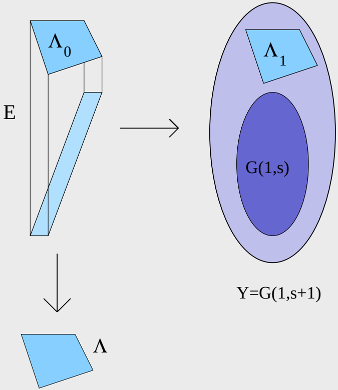

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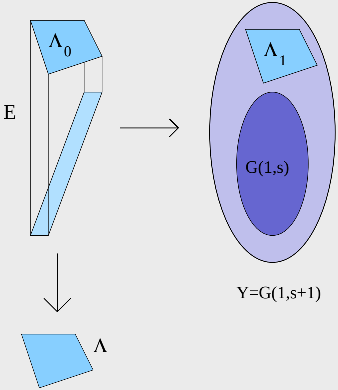

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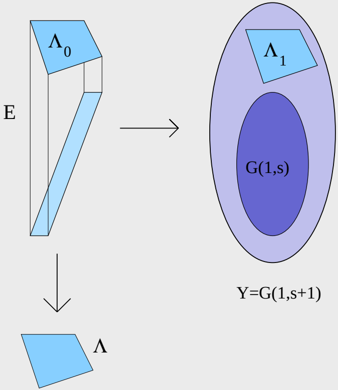

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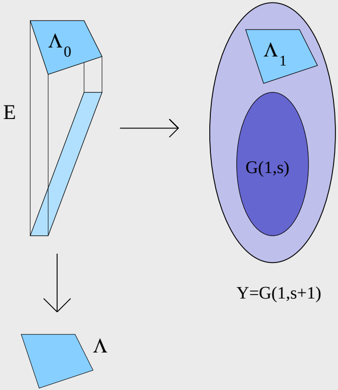

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$E = \mathbb{P}_\Lambda(\Omega(1) \oplus \mathscr{O})$.
E is the blow-up of $\mathbb{G}(1, s+1)$ along a subgrassmannian $\mathbb{G}(1, s)$.

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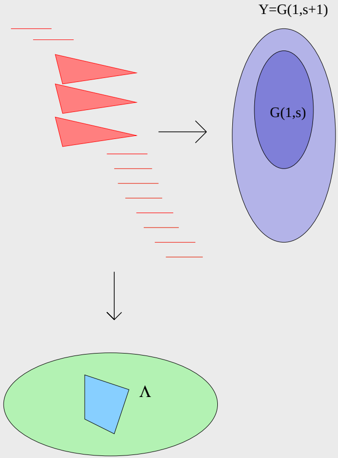

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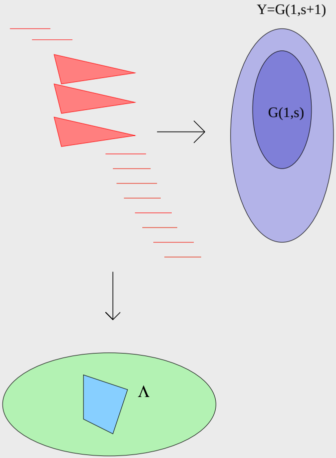
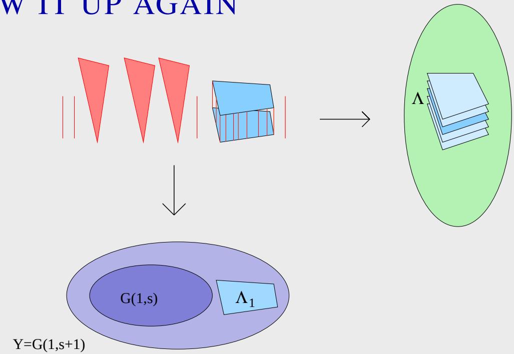


$\widetilde{\pi}$ is a scroll over $\mathbb{G}(1, s+1)$. Not enough information to describe completely $\widetilde{X}$.

Blow it up again



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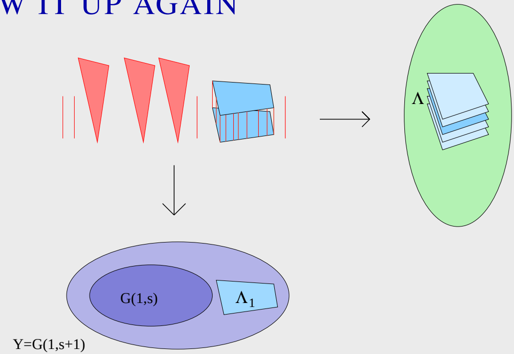

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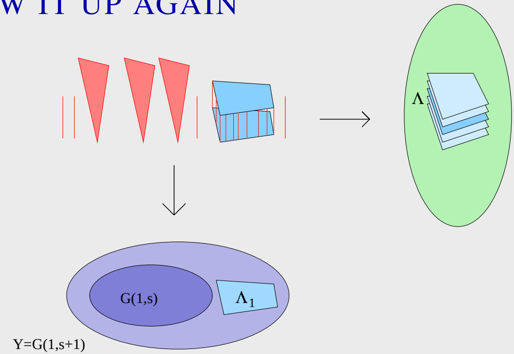

We find a subvariety $\Sigma \subset X$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^s$, containing Λ. The blow-up of X along Σ is isomorphic to the blow-up of a hyperplane section of $\mathbb{G}(1, s+2)$ along the subvariety parametrizing lines in $\mathbb{P}^{s+2}$ meeting a given line.

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$\varphi\colon X \to C$ is an extremal contraction and every fiber is a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$.

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$l_2 = \pi'(l'_2)$ line in a fiber of the projection of onto the second factor.