Rational curves and bounds on the Picard number of Fano manifolds

Gianluca Occhetta
(joint work with Carla Novelli)

Projective Algebraic Geometry in Milano,
June 2009

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Moreover X has an elemenary fiber type contraction.

Families of rational curves

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$$\begin{array}{ccc} U & \xrightarrow{i} & X \\ {\scriptstyle\pi}\downarrow & & \\ V & & \end{array}$$

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$[V]$ is the numerical class of a curve parametrized by V.

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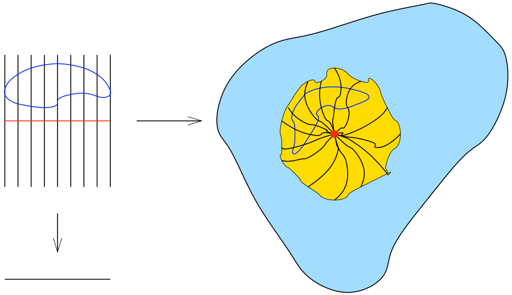

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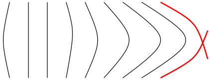

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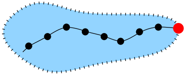

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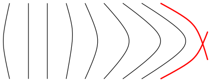

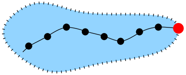

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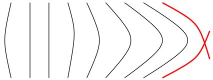

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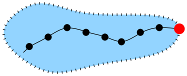

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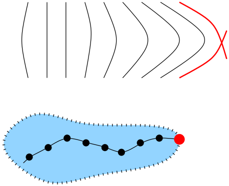

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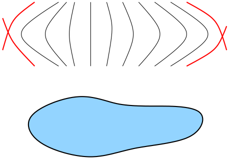

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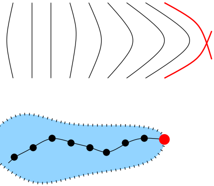

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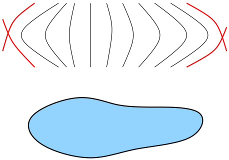

If all the irreducible components of reducible cycles are numerically proportional we say that $\mathscr{V}$ is quasi-unsplit.

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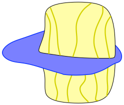

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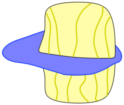

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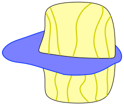

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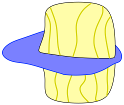

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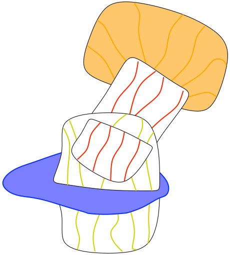

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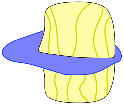

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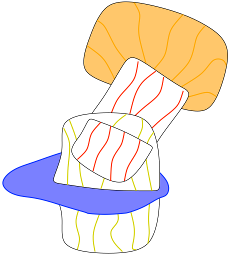

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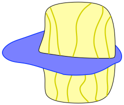

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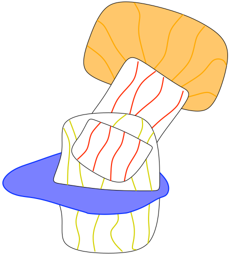

Notice that $\mathrm{Locus}(V)_x = \mathrm{Locus}(V_x)$.

Analogously we define $\mathrm{Locus}(\mathscr{W}^1, \ldots, \mathscr{W}^k)_Y$ for Chow families $\mathscr{W}^1, \ldots, \mathscr{W}^k$ of rational 1-cycles.

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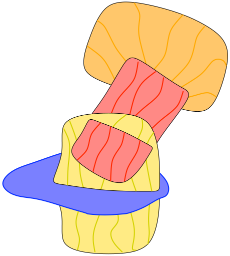

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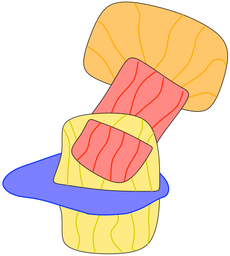

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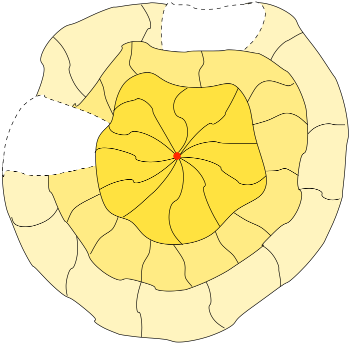

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Every curve contained in $\mathrm{Locus}(\mathscr{V})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve in Y and of irreducible components of cycles parametrized by $\mathscr{V}$ which meet Y.

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In particular, if $\mathscr{V}^1, \dots, \mathscr{V}^k$ are quasi-unsplit families, then $\rho_X \leq k$

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The families $V^1, \dots, V^k$ are locally unsplit and numerically independent.

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The last assertion follows from

Theorem (_)

A smooth complex projective variety X of dimension n is isomorphic to $\mathbb{P}^{n(1)} \times \cdots \times \mathbb{P}^{n(k)}$ iff $\exists V^1, \ldots, V^k$ unsplit and covering with $\sum -K_X \cdot V^k = n + k$ such that $\dim \langle [V^1], \ldots, [V^k] \rangle = k$ in $N_1(X)$.

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If for some j the family V^j is not unsplit we have $-K_X \cdot V^j \geq 2i_X$ so this can happen for at most one j and implies $k = 1$.

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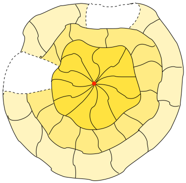

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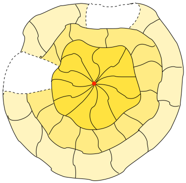

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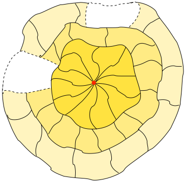

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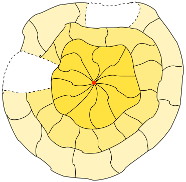

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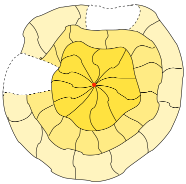

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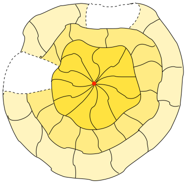

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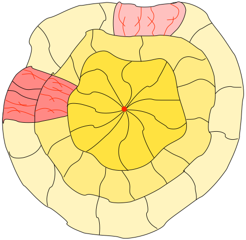

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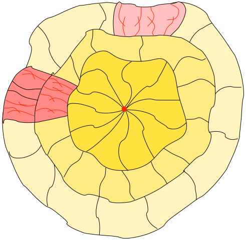

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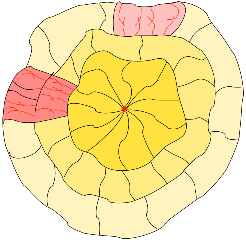

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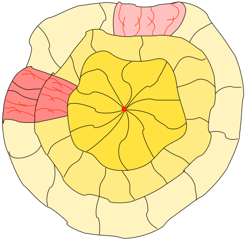

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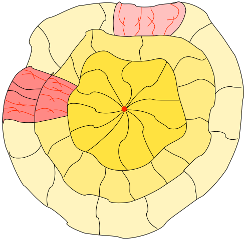

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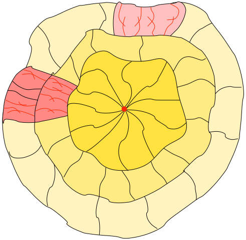

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