SOME APPLICATIONS OF THE TRACE CONDITION FOR PLURIHARMONIC FUNCTIONS IN $\mathbf{C}^n$

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Abstract. In this paper we investigate some applications of the trace condition for pluriharmonic functions on a smooth, bounded domain in $\mathbf{C}^n$. This condition, related to the normal component on ∂D of the $\overline{\partial}$-operator, permits us to study the Neumann problem for pluriharmonic functions and the $\overline{\partial}$-problem for $(0,1)$-forms on D with solutions having assigned real part on the boundary.

1. Introduction

Let D be a smoothly bounded domain in $\mathbf{C}^n$. We are interested in some results that can be obtained from the trace condition for pluriharmonic functions introduced by Fichera in the papers [F1],[F2],[F3] and investigated in [P]. This condition has a global character: a real function on ∂D is the trace of a pluriharmonic function on D if it is orthogonal, in the $L^2(\partial D)$-norm, with respect to a suitably chosen space of functions. This approach is alternative to the local study of tangential differential conditions (see for example [R]§18.3 and [F3] and the references given there).

In this paper we first consider the Neumann problem for pluriharmonic functions. Let $\lambda > 0$. Given a real function ϕ on the boundary, of class C^λ, we show (Theorem 1) that the solutions of the classical Neumann problem

$$\frac{\partial U}{\partial \nu} = \phi \quad \text{on } \partial D$$

are pluriharmonic on D if and only if ϕ is orthogonal in $L^2(\partial D)$ to the subspace of real harmonic functions that admit a decompostion $H_1 + iH_2$ with $H_1, H_2 \in Harm^1_0(D)$ (see §§2,3 for the precise definitions). When the domain is the unit ball of $\mathbf{C}^n$, this result can be expressed in terms of spherical harmonics. We thus obtain (Corollary 1) another proof of a theorem given by Dzhuraev in [D2]: U is pluriharmonic if and only if it satisfies the Gauss compatibility condition $\int_S \phi d\sigma = 0$ and the trace condition.

In §4 we investigate the $\overline{\partial}$-problem for $(0,1)$-forms on D with a boundary condition. Given a $\overline{\partial}$-closed form $f \in C^0_{0,1}(\overline{D})$ and a real C^λ function g on ∂D, we look for a solution $u \in C^1(\overline{D})$ of the problem

$$\overline{\partial} u = f \quad \text{on } D, \quad \operatorname{Re} u = g \quad \text{on } \partial D$$

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We prove (Theorem 2) that a solution exists if and only if $\int_{\partial D} g\ \overline{\partial}_n H d\sigma$ equals the real part of $\int_D \overline{f} \wedge *\overline{\partial} H$ for every $H \in Harm^1_0(D)$. For this problem too, we can rewrite the compatibility condition on the unit ball in terms of spherical harmonics (Corollary 2). A similar approach to this problem on the ball of $\mathbf{C}^n$ appears in [D1].

2. Notations and preliminaries

2.1. Let $D = \{z \in \mathbf{C}^n : \rho(z) < 0\}$ be a bounded domain in $\mathbf{C}^n$ with boundary of class $C^m, m \geq 1$. We assume $\rho \in C^m$ on $\mathbf{C}^n$ and $d\rho \neq 0$ on ∂D.

For every $\alpha, 0 \leq \alpha \leq m$, we denote by $Ph^\alpha(D)$ the space of real pluriharmonic functions of class $C^\alpha(\overline{D})$ and similarly for holomorphic functions $A^\alpha(D)$ and complex harmonic functions $Harm^\alpha(D)$. We denote by $Ph^\alpha_{\partial D}(D)$ the space of restrictions to ∂D of pluriharmonic functions in $Ph^\alpha(D)$ and by $ReA^\alpha(D)$ the space of real parts of element of $A^\alpha(D)$.

2.2. For every $F \in C^1(\overline{D})$, in a neighbourhood of ∂D we have the decomposition of $\overline{\partial} F$ in the tangential and the normal parts

$$\overline{\partial} F = \overline{\partial}_b F + \overline{\partial}_n F \frac{\overline{\partial}\rho}{|\overline{\partial}\rho|}$$

where $\overline{\partial}_n F = \sum_k \dfrac{\partial F}{\partial \overline{\zeta}_k} \dfrac{\partial \rho}{\partial \zeta_k} \dfrac{1}{|\overline{\partial}\rho|}$.

If ν denotes the outer unit normal to ∂D and $\tau = i\nu$, then we can also write $\overline{\partial}_n F = \dfrac{1}{2}\left(\dfrac{\partial F}{\partial \nu} + i \dfrac{\partial F}{\partial \tau}\right)$. The normal part of $\overline{\partial} F$ on ∂D can also be expressed by means of the Hodge $*$-operator and the Lebesgue surface measure $d\sigma$: $\overline{\partial}_n F d\sigma = *\overline{\partial} F_{|\partial D}$ (see [K]§§3.3 and 14.2).

2.3. We shall denote by $Harm^1_0(D)$ the real subspace of $Harm^1(D)$

$$Harm^1_0(D) = \{H \in Harm^1(D) : \overline{\partial}_n H \text{ is real on } \partial D\}$$

This space can be characterized in terms of Bochner-Martinelli operator M. In [P]§4 it was shown that $F \in Harm^1_0(D)$ if and only if $\operatorname{Im} M(F) = \operatorname{Im} F$ in D. We recall the integral orthogonality condition that characterizes the traces on ∂D of pluriharmonic functions on D which was introduced (in a different form) by Fichera in [F1] (see also [F2],[F3]) and that was studied in [P]:

$$(\star) \qquad \int_{\partial D} U\ \overline{\partial}_n H d\sigma = 0$$

for every $H \in Harm^1_0(D)$.

It was shown in [P] that this condition is necessary for pluriharmonicity when ∂D is of class C^1 and $U \in C^\alpha(\overline{D})$, $\alpha > 0$, or when ∂D is of class C^2 and $U \in C^0(\overline{D})$. The trace condition is sufficient in the case when $U \in C^{1+\lambda}$, $\lambda > 0$. If the boundary value u is only continuous, the same result holds on strongly pseudoconvex domains and on weakly pseudoconvex domains with real analytic boundary.

Remark. When $n = 1$ and $H^1(D, \mathbf{R}) = 0$ condition $(\star)$ is void, since H belongs to $Harm^1_0(D)$ if and only if H is holomorphic on D (cf. [P]).

In general ($n \geq 1$), the space of $C^1(\overline{D})$-holomorphic functions on D is the maximal complex subspace in $Harm^1(D)$ that contains $Harm^1_0(D)$. This follows from a theorem of Kytmanov and Aizenberg [KA] (cf. also [K]§§14 and 15): a $C^1(\overline{D})$-harmonic function F is holomorphic on D if and only if $\overline{\partial}_n F = 0$ on ∂D.

2.4. Let B be the unit ball of $\mathbf{C}^n$ and let $S = \partial B$. The space $L^2(S)$ is the direct sum of pairwise orthogonal spaces $H(s,t), s \geq 0, t \geq 0$, where $H(s,t)$ is the space of harmonic homogeneous polynomials of total degree s in z and total degree t in $\overline{z}$ (see [R] §12). These spaces are the eigenspaces of the Bochner-Martinelli operator.

In this case, the trace condition for pluriharmonic functions reduces to the orthogonality to the spaces $H(s,t)$, $s,t > 0$. This is the content of a theorem of Nagel and Rudin [NR] (see also [P]§5).

3. The Neumann problem for pluriharmonic functions

3.1. In this section we study the Neumann problem for pluriharmonic functions: given ϕ on the boundary ∂D, find necessary and sufficient conditions for the existence of a pluriharmonic function U on D such that $\dfrac{\partial U}{\partial \nu} = \phi$ on ∂D.

We start from the following result announced by Fichera in [F1],[F2],[F3]. Let D be a simply connected domain, with boundary of class $C^{1+\lambda}$, $\lambda > 0$. Let A, B, C be real functions of class $C^1(\overline{D})$, harmonic on D, such that

$$\frac{\partial A}{\partial \tau} + \frac{\partial B}{\partial \nu} = 0, \qquad \frac{\partial A}{\partial \nu} - \frac{\partial C}{\partial \tau} = 0$$

on ∂D. Given $\phi \in C^\lambda(\partial D)$, there exists a pluriharmonic function $U \in C^1(\overline{D})$ such that $\dfrac{\partial U}{\partial \nu} = \phi$ on ∂D if and only if

$$\int_{\partial D} \phi(B - C) d\sigma = 0$$

for any triplet A, B, C.

The preceding condition is equivalent to the following

$$(\star\star) \qquad \int_{\partial D} \phi(H_1 + iH_2) d\sigma = 0$$

for every pair $H_1, H_2 \in Harm^1_0(D)$ such that $H_1 + iH_2$ is real.

This can be seen setting $H_1 = -C + iA$, $H_2 = -A - iB$. Then $H_1, H_2 \in Harm^1_0(D)$ and $H_1 + iH_2 = B - C$.

Theorem 1. *Let $H^1(D, \mathbf{R}) = 0$ and ∂D of class $C^{1+\lambda}$. Let $\phi \in C^\lambda(\partial D)$ be a real function, with $\lambda > 0$. Then there exists $U \in Ph^1(D)$ such that $\dfrac{\partial U}{\partial \nu} = \phi$ on ∂D if and only if ϕ satisfies condition $(\star\star)$.*

Proof. We relate the condition $(\star\star)$ to the trace condition $(\star)$. Let H_1, H_2 be as before. From the complex version of Green's formula (see for example [K]§11.3) we get

$$\int_{\partial D} U(\overline{\partial}_n H_1 + i\overline{\partial}_n H_2) d\sigma = \int_{\partial D} \partial_n U(H_1 + iH_2) d\sigma$$

for every real harmonic function $U \in C^1(\overline{D})$. Taking real parts, we obtain

$$\int_{\partial D} U\overline{\partial}_n H_1 d\sigma = \frac{1}{2}\int_{\partial D} \frac{\partial U}{\partial \nu}(H_1 + iH_2)d\sigma$$

Now assume that ϕ satisfies condition $(\star\star)$. When $H_1 = 1, H_2 = 0$ $(\star\star)$ becomes the Gauss compatibility condition for the classical Neumann problem for harmonic functions. Let λ' be any positive number smaller than λ. Let $U \in Harm^{1+\lambda'}(D)$ be real, such that $\dfrac{\partial U}{\partial \nu} = \phi$ on ∂D (determined up to an additive constant). We show that U satisfies the trace condition. If $H \in Harm_0^{1+\lambda'}(D)$, we can set $H_1 = H$, $H_2 = -\operatorname{Im} H + iG$, where $G \in C^1(\overline{D})$ is a real harmonic solution of $\dfrac{\partial G}{\partial \nu} = \dfrac{\partial \operatorname{Im} H}{\partial \tau}$ on ∂D. Since $H_1, H_2 \in Harm_0^1$ and $H_1 + iH_2$ is real, the integral $\int_{\partial D} \phi(H_1 + iH_2)d\sigma = 2\int_{\partial D} U\overline{\partial}_n H d\sigma$ vanishes. From Theorem 2 in [P] we get that U is the real part of a holomorphic function. Note that in the proof of the cited theorem it is sufficient to consider functions $H \in Harm_0^{1+\lambda'}(D)$.

Conversely, if U is pluriharmonic on D, then Theorem 1 in [P] says that U satisfies condition $(\star)$ and then $\int_{\partial D} \phi(H_1 + iH_2)d\sigma = 0$.

Remarks. (i) In effect the proof shows that the condition $(\star\star)$ implies that $U \in ReA^{1+\lambda'}(D)$ even without the topological assumption on D.

(ii) If $n = 1$, H_1, H_2 belong to $Harm_0^1(D)$ if and only if they are holomorphic functions on D. Then $H_1 + iH_2$ is a real valued holomorphic function, that is a constant. As is expected, condition $(\star\star)$ reduces to the Gauss compatibility condition for the Neumann problem.

3.2. If B is the unit ball of $\mathbf{C}^n$ and S the unit sphere, the preceeding result can be rewritten in terms of the spaces of harmonic homogeneous polynomials $H(s,t)$. Let N_0 be the real linear projection in $L^2(S)$ introduced in [P]. It is defined for $P_{s,t} \in H(s,t)$ by

$$N_0(P_{s,t}) = \begin{cases} \dfrac{s}{s+t}P_{s,t} + \dfrac{t}{s+t}\overline{P_{s,t}}, & \text{for } t > 0 \\ P_{s,t}, & \text{for } t = 0 \end{cases}$$

The space $Harm_0^1(B)$ coincides with $\operatorname{Fix}(N_0) = \{F \in Harm^1(B) : N_0(F) = F\}$. We show that Theorem 1 in the case of the ball reduces to a result given by Dzhuraev in [D2].

Corollary 1. *Let $\phi \in C^\lambda(S)$ be a real function, with $\lambda > 0$. Then there exists $U \in C^1(\overline{B})$, pluriharmonic on B and such that $\dfrac{\partial U}{\partial \nu} = \phi$ on S if and only if $\int_S \phi d\sigma = 0$ and ϕ is orthogonal to the spaces $H(s,t)$ in $L^2(S)$ for any $s,t > 0$.*

Proof. If $s,t > 0$, we set $H_1 = N_0(P_{s,t})$ and $H_2 = N_0(-\operatorname{Im} H_1)$. Note that for $s,t > 0$ we have $N_0(\operatorname{Re} F) = N_0(F)$, $N_0(\operatorname{Im} F) = N_0(-iF)$. An easy computation gives $H_1 + iH_2 = \frac{4st}{(s+t)^2}\operatorname{Re} P_{s,t}$. Replacing $P_{s,t}$ with $iP_{s,t}$, we get $H_1 + iH_2 = \frac{4st}{(s+t)^2}\operatorname{Im} P_{s,t}$. Then the condition $(\star\star)$ is equivalent to the orthogonality of ϕ to the spaces $H(s,t)$ and Theorem 1 gives the result.

4. $\overline{\partial}$-problem with assigned real part on the boundary

4.1. In this section we study the $\overline{\partial}$-problem for $(0,1)$-forms on D with a boundary condition. Let $f \in C^0_{0,1}(\overline{D})$ be a $\overline{\partial}$-closed form with continuous coefficients on $\overline{D}$ and let g be a real continuous function on ∂D. We look for a function $u \in C^1(\overline{D})$ such that

$$\overline{\partial} u = f \quad \text{on } D, \quad \operatorname{Re} u = g \quad \text{on } \partial D$$

The solution, if it exists, is unique up to an imaginary constant. This problem was considered by Dzhuraev in [D1] and [D2] in the case of the unit ball.

If $H^1(D,\mathbf{R}) = 0$ and there exists a solution w of $\overline{\partial} w = f$ which is continuous on $\overline{D}$, the problem can be reduced to the trace condition for pluriharmonic functions, since then it amounts to finding a holomorphic function h on D such that $\operatorname{Re}(h + w) = g$ on ∂D. If ∂D is of class C^2 and D is strongly pseudoconvex, there exists a solution operator $S_q : C_{0,q}(\overline{D}) \to C_{0,q-1}(D)$ such that if $\overline{\partial} f = 0$ and f is of class $C^k(\overline{D})$, then $\overline{\partial} S_q(f) = f$ and $S_q(f) \in C^{k+1/2}_{0,q-1}(\overline{D})$ (see for example [RA] and the references given there). If ∂D is of class C^∞ and D is strongly pseudoconvex, another well-known solution operator is given by the Neumann operator N related to $\Box$. If $\overline{\partial} f = 0$, then $\overline{\partial}^* N(f)$ is the unique solution of minimal $L^2(D)$-norm of the equation $\overline{\partial} u = f$.

Theorem 2. *Let D be a smoothly bounded strongly pseudoconvex domain. Assume that $H^1(D,\mathbf{R}) = 0$. Let $f \in C^0_{0,1}(\overline{D})$ and let g be a real C^λ function on ∂D ($\lambda > 0$). Then there exists $u \in C^1(\overline{D})$ such that $\overline{\partial} u = f$ on D, $\operatorname{Re} u = g$ on ∂D if and only if $\overline{\partial} f = 0$ and for every $H \in Harm^1_0(D)$*

$$\int_{\partial D} g \ \overline{\partial}_n H d\sigma = \operatorname{Re} \int_D \overline{f} \wedge * \overline{\partial} H$$

Proof. Let $\alpha < \lambda$ be a positive number with $\alpha < \frac{1}{2}$. The function $g - \operatorname{Re} S_1(f) \in C^\alpha(\partial D)$ is the trace of a pluriharmonic function if and only if it satisfies the trace condition ($\star$). This follows from Theorems 1 and 3 in [P]. For $H \in Harm^1_0(D)$, we transform the integral on ∂D involving f in an integral on D:

$$\int_{\partial D} \overline{S_1(f)} \ \overline{\partial}_n H d\sigma = \int_{\partial D} \overline{S_1(f)} * \overline{\partial} H = \int_D \overline{\overline{\partial} S_1(f)} \wedge * \overline{\partial} H = \int_D \overline{f} \wedge * \overline{\partial} H$$

Here we have used the fact that $*\overline{\partial} H$ is a closed $(n-1,n)$-form, since $*\partial(*\overline{\partial} H) = -\overline{\partial}^* \overline{\partial} H = -\Box H = 2\Delta H = 0$.

The last integral is the Hodge product $(\overline{f}, \partial \overline{H})$ on D. Since $\overline{\partial}_n H$ is real on ∂D, the trace condition becomes

$$0 = \int_{\partial D} (g - \operatorname{Re} S_1(f)) \ \overline{\partial}_n H d\sigma = \int_{\partial D} g \ \overline{\partial}_n H d\sigma - \operatorname{Re} \int_{\partial D} \overline{S_1(f)} \ \overline{\partial}_n H d\sigma =$$

$$= \int_{\partial D} g \ \overline{\partial}_n H d\sigma - \operatorname{Re} \int_D \overline{f} \wedge * \overline{\partial} H$$

Note that $Ph^\alpha(D) = ReA^\alpha(D)$, since the first derivatives of $U \in Ph^\alpha(D)$ satisfy a Hardy-Littlewood estimate (see [L] §2 for example) and then the same holds for the harmonic conjugate.

4.2. If D is the unit ball, the condition given in the theorem has the following more explicit form.

Corollary 2. *There exists $u \in C^1(\overline{B})$ such that $\overline{\partial} u = f$ on B, $\operatorname{Re} u = g$ on S if and only if $\overline{\partial} f = 0$ and for every $s, t > 0$ and $P_{s,t} \in H(s,t)$*

$$\operatorname{Re} \int_S g P_{s,t} d\sigma = \operatorname{Re} \int_B \sum_{k=1}^n \overline{f_k} \frac{\partial}{\partial \overline{z}_k} \left(\frac{P_{s,t}}{t} + \frac{\overline{P_{s,t}}}{s} \right) dv$$

$$\operatorname{Im} \int_S g P_{s,t} d\sigma = \operatorname{Im} \int_B \sum_{k=1}^n \overline{f_k} \frac{\partial}{\partial \overline{z}_k} \left(\frac{P_{s,t}}{t} - \frac{\overline{P_{s,t}}}{s} \right) dv$$

Proof. For $H = N_0(P_{s,t})$ the left integral in Theorem 2 becomes $\int_S g \frac{2st}{s+t} \operatorname{Re} P_{s,t} d\sigma$, while the right integrand $\overline{f} \wedge * \overline{\partial} N_0(P_{s,t})$ is equal to

$$\sum_k \overline{f}_k dz_k \wedge \frac{2}{s+t} \left(\frac{i}{2} \right)^n \sum_k (-1)^{k-1} \frac{\partial}{\partial \overline{z}_k} (s P_{s,t} + t \overline{P_{s,t}}) dz[k] \wedge \overline{dz}$$

Since $dv = \left(\frac{i}{2} \right)^n dz \wedge \overline{dz}$, we get the first condition. Replacing $P_{s,t}$ with $i P_{s,t}$ we get the second one.

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