Quaternionic regular functions and the $\overline{\partial}$–Neumann problem in $\mathbb{C}^2$

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1. Some notations

$$\mathbb{C}^2 \ni (z_1, z_2) = (x_0 + ix_1, x_2 + ix_3) \leftrightarrow$$
$$q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$$

Let Ω be a bounded domain in $\mathbb{H} \approx \mathbb{C}^2$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is (left) *regular* on Ω if

$$\mathcal{D}f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0 \text{ on } \Omega,$$

f is (left) ψ*-regular* on Ω if

$$\mathcal{D}'f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \text{ on } \Omega.$$

Remarks 1. f is ψ-regular $\Leftrightarrow$

$$\frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \quad \frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial \overline{f_2}}{\partial z_1} \;\Leftrightarrow$$

$$*\overline{\partial} f_1 = -\frac{1}{2}\partial(\overline{f_2} d\bar{z}_1 \wedge d\bar{z}_2).$$

2. Every regular or ψ-regular function is harmonic.
3. Every holomorphic map (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$.
4. If Ω is pseudoconvex, every complex harmonic function f_1 is the complex component of a ψ-regular function f on Ω.

2. Main results

2.1 A differential criterion for regularity

Theorem 1. *$f = f_1 + f_2 j \in C^1(\overline{\Omega})$ is ψ-regular on Ω if and only if f is harmonic on Ω and*

$$(\overline{\partial}_n - jL)f = 0 \quad \textit{on } \partial\Omega. \qquad (*)$$

□

$\overline{\partial}_n f$ is the normal component of $\overline{\partial} f$ on $\partial\Omega$, defined by: $\overline{\partial}_n f d\sigma = *\overline{\partial} f_{|\partial\Omega}$,

L is the tangential Cauchy-Riemann operator

$$L = \frac{1}{|\overline{\partial}\rho|}\left(\frac{\partial\rho}{\partial\bar{z}_2}\frac{\partial}{\partial\bar{z}_1} - \frac{\partial\rho}{\partial\bar{z}_1}\frac{\partial}{\partial\bar{z}_2}\right).$$

Remark. Condition (*) generalizes both the CR-tangential equation $L(f) = 0$ and the condition $\overline{\partial}_n f = 0$ on $\partial\Omega$ that distinguishes holomorphic functions among complex harmonic functions (Aronov and Kytmanov).

The single equation (*) is equivalent to the following system of complex equations on $\partial\Omega$:

$$\overline{\partial}_n f_1 = -\overline{L(f_2)} \qquad (\mathrm{C}_1)$$

$$\overline{\partial}_n f_2 = \overline{L(f_1)} \qquad (\mathrm{C}_2)$$

A weak version of Theorem 1 gives a trace theorem:

Theorem 2. *A continuous function $f : \partial\Omega \to \mathbb{H}$ is the trace of a ψ-regular function on Ω if and only if it satisfies the integral condition*

$$\int_{\partial\Omega} \bar{f}\left(\overline{\partial}_n - jL\right)\phi \, d\sigma = 0 \;\forall\, \phi \in Harm^1(\overline{\Omega}). \quad \square$$

From Theorem 1 we immediately get the following result about regular functions:

Theorem 3. $f = f_1 + f_2 j \in C^1(\overline{\Omega})$ *is regular on* Ω *if and only if* f *is harmonic on* Ω *and*

$$(N - jT)f = 0 \text{ on } \partial\Omega, \text{ where}$$

$$N = \frac{\partial\rho}{\partial z_1}\frac{\partial}{\partial\bar{z}_1} + \frac{\partial\rho}{\partial\bar{z}_2}\frac{\partial}{\partial z_2},\ T = \frac{\partial\rho}{\partial z_2}\frac{\partial}{\partial\bar{z}_1} - \frac{\partial\rho}{\partial\bar{z}_1}\frac{\partial}{\partial z_2}. \quad \square$$

2.2 A criterion for holomorphicity

When $\partial\Omega$ is connected, Hartogs Theorem can be applied to improve the previous results. Now conditions

$$\overline{\partial}_n f_1 = -\overline{L(f_2)} \qquad (\mathrm{C}_1)$$

$$\overline{\partial}_n f_2 = \overline{L(f_1)} \qquad (\mathrm{C}_2)$$

are equivalent: one of them implies the ψ-regularity of f.

Remark. The connectedness of $\partial\Omega$ is a necessary assumption: consider a locally constant function on $\partial\Omega$.

The equivalence of C_1 and C_2 can be used to get the following criterion for holomorphicity:

Theorem 4. *Let $\Omega \subseteq \mathbb{C}^2$ be bounded, with connected boundary $\partial\Omega$. Let $a \in \mathbb{C}$. If $h \in C^1(\overline{\Omega})$ is complex harmonic and satisfies the condition $\overline{\partial}_n h = \overline{aL(h)}$ on $\partial\Omega$, then h is holomorphic on Ω.* $\square$

Remark. The case $a = 0$ is a theorem of Aronov and Kytmanov. Mixed differential conditions of this type have been studied in particular by Chirka and Kytmanov.

2.3 Regularity and the $\overline{\partial}$-Neumann problem

The $\overline{\partial}$-Neumann for complex functions can be formulated in the following way:

$$\overline{\partial}_n g = \phi \text{ on } \partial\Omega, \quad g \text{ harmonic in } \Omega,$$

with compatibility condition

$$\int_{\partial\Omega} \phi \overline{h} d\sigma = 0 \quad \forall h \in \mathcal{O}(\overline{\Omega}).$$

If $\partial\Omega$ is connected and C^∞–smooth and Ω is strongly pseudoconvex or weakly pseudoconvex with real analytic boundary, the solvability of $\overline{\partial}$-Neumann problem (Kytmanov) applied to the equation

$$\overline{\partial}_n f_2 = \overline{L(f_1)} \tag{C$_2$}$$

allows to achieve the following:

Theorem 5. *Let $f_1 : \partial\Omega \to \mathbb{C}$ be of class C^∞. Then f_1 is the trace on $\partial\Omega$ of one complex component of a ψ-regular function f on Ω, of class C^∞ on $\overline{\Omega}$.* □

Remark. f_2 is determined up to a holomorphic function, so f is uniquely determined by the orthogonality condition

$$\int_{\partial\Omega} (f - f_1)\overline{h} d\sigma = 0 \quad \forall h \in \mathcal{O}(\overline{\Omega}).$$

This defines a $\mathbb{C}$-linear operator

$$R : C^\infty(\partial\Omega) \to M^\infty(\Omega).$$

Corollary 1. *Let $M^\infty(\Omega)$ be the right $\mathbb{H}$-module of left ψ-regular functions of class C^∞ on $\overline{\Omega}$. The mapping C defined by $C(f) = f_{1|\partial\Omega}$ for every $f = f_1 + f_2 j \in M^\infty(\Omega)$ induces an isomorphism of real spaces*

$$\frac{M^\infty(\Omega)}{A^\infty(\Omega, \mathbb{C}^2)} \xrightarrow{\approx} \frac{C^\infty(\partial\Omega)}{CR(\partial\Omega)}.$$

□

2.4 An application: a product in $M^\infty(\Omega)$

The existence of a right inverse for C

$$M^\infty(\Omega) \underset{C}{\overset{R}{\rightleftarrows}} C^\infty(\partial\Omega) \Longleftrightarrow C \circ R = Id_{C^\infty(\partial\Omega)}$$

allows to define a product in $M^\infty(\Omega)$, with respect to which $M^\infty(\Omega)$ becomes a *commutative* $\mathbb{R}$-algebra, with unity the constant function 1, and which contains $A^\infty(\Omega, \mathbb{C}^2)$ as a subalgebra with respect to the product

$$(f_1, f_2) \cdot (g_1, g_2) = (f_1 g_1 + f_2 g_2, f_1 g_2 + f_2 g_1).$$

Given $f, g \in M^\infty(\Omega)$, let

$$f * g = R(f_1 g_1) - (f - R(f_1)) j (g - R(g_1))$$

where $f_1 = C(f)$, $g_1 = C(g)$.

Let $\phi : M^\infty(\Omega) \to M^\infty(\Omega)$

$$\phi(f) = f(1 + j).$$

The product $m_\Omega(f, g)$ can be defined as

$$m_\Omega(f, g) = \phi^{-1}(\phi(f) * \phi(g)).$$

3. The case of the unit ball

When $\Omega = B$ is the unit ball in $\mathbb{C}^2$, S the unit sphere, the operators

$$\overline{\partial}_n = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}, \quad L = z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2}$$

preserve harmonicity. Condition (*) in Theorem 1 can be reformulated for polynomials. Let

$$D_k = \sum_{0 \le l \le k/2-1} \frac{(k-2l-1)!(2l-1)!!}{k!(l+1)!} 2^l \Delta^{l+1}.$$

Theorem 6. *The restriction to S of a homogeneous polynomial $f = f_1 + f_2 j$ of degree k extends as a ψ-regular function into B if and only if*

$$(\overline{\partial}_n - D_k) f_1 + \overline{L(f_2)} = 0 \quad \textit{on } S.$$

It extends as a regular function if and only if

$$(N - D_k) f_1 + \overline{T(f_2)} = 0 \quad \textit{on } S.$$

Theorem 5 has the following homogeneous version:

Theorem 7. *a) For every $f_1 \in \mathcal{P}_k$ (complex k-homogeneous polynomial), there exists $f_2 \in \mathcal{P}_k$ such that the trace of $f = f_1 + f_2 j$ on S extends as a ψ-regular polynomial of degree $\leq k$ on $\mathbb{H}$.*
b) If f_1 is harmonic, then f belongs to the right $\mathbb{H}$-module U_k^{ψ} of ψ-regular homogeneous polynomials of degree k.

The right inverse

$$R : \mathcal{H}_k(S) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(S) \to U_k^{\psi}$$

of C ($\mathcal{H}_{p,q}$ the space of harmonic homogeneous polynomials of degree p in z and q in $\bar{z}$, $\mathcal{H}_k(S)$ the space of *spherical harmonics*) gives the following:

Corollary 2. *The restriction first-component operator C induces isomorphisms*

$$\frac{U_k^{\psi}}{\mathcal{H}_{k,0} + \mathcal{H}_{k,0} j} \simeq \frac{\mathcal{H}_k(S)}{\mathcal{H}_{k,0}(S)}.$$

These isomorphisms can be applied to obtain $\mathbb{H}$-bases for U_k^ψ starting from $\mathbb{C}$-bases of $\mathcal{H}_{p,q}$ $(p+q=k)$. This construction preserves orthogonality w.r.t. $L^2(S)$.

Given bases $\{P_l\}$ of $\mathcal{H}_{p,q}$, a suitably chosen subset of the images

$$R(P_l) = \begin{cases} P_l & \text{if } q = 0 \\ P_l + \dfrac{1}{p+1}\overline{L(P_l)}j & \text{if } q > 0 \end{cases}$$

gives a $\mathbb{H}$-basis for U_k^ψ ($\dim_{\mathbb{H}} U_k^\psi = \frac{(k+1)(k+2)}{2}$).

A possible choice for a $L^2(S)$-orthogonal basis of $\mathcal{H}_{p,q}$ is given by the $p+q+1$ polynomials

$$P_l(z_1,z_2) = \sum_{r=max\{0,l-p\}}^{min\{q,l\}} c_{l,r}{z_1}^{p-l+r}{z_2}^{l-r}\bar{z}_1^r\bar{z}_2^{q-r}$$

where $c_{l,r} = (-1)^r\binom{p}{l-r}\binom{q}{r}$ and $l = 0,\ldots,p+q$.

Cf. **RegularHarmonics**: a *Mathematica* 4.2 package available at
`www.science.unitn.it`
`/~perotti/regular harmonics.htm`

4. Sketch of proofs

4.1 Theorem 1 (criterion for ψ-regularity)

The main point is a property of the differential form associated to the Cauchy-Fueter kernel for ψ-regular functions: its first complex component is the Bochner-Martinelli kernel in dimension 2 (Fueter–Vasilevski–Shapiro).

We show that the Bochner-Martinelli integral representation formula for harmonic functions, under condition (*), is the same as the Cauchy-Fueter integral representation formula, from which regularity follows.

4.2 Theorem 2 (trace theorem)

The result follows from the jump formula for the Cauchy-Fueter integral. Using again the property above, we show that the Cauchy-Fueter integral of $f \in C(\partial\Omega)$ vanishes on the complement $\mathbb{C}^2 \setminus \overline{\Omega}$ under condition

$$\int_{\partial\Omega} \bar{f}\left(\overline{\partial}_n - jL\right)\phi \; d\sigma = 0 \;\forall\; \phi \in Harm^1(\overline{\Omega}).$$

When $\partial\Omega$ is *connected* and one of conditions C_1, C_2 (say C_2) holds, the Cauchy-Fueter integral of f defines on $\mathbb{C}^2 \setminus \overline{\Omega}$ a *complex valued* ψ-regular function F^- $\Rightarrow$ a holomorphic function on $\mathbb{C}^2 \setminus \overline{\Omega}$ $\Rightarrow$ a holomorphic function $\tilde{F}^-$ on $\mathbb{C}^2$.

In this way we get a ψ-regular function $F = F^+ - \tilde{F}^-_{|\Omega}$ on Ω, whose trace on $\partial\Omega$ is f.

4.3 Theorem 4 (criterion for holomorphicity)

Given $f = ah + hj$, condition C_2 is satisfied, and then f is ψ-regular. From ψ-regularity equations we obtain

$$\overline{\partial} h = 0.$$

4.4 Theorem 5 ($\overline{\partial}$-Neumann problem)

The result follows easily since $\phi = \overline{L(f_1)}$ satisfies the compatibility condition for $\overline{\partial}$-Neumann problem. Then there exists f_2 such that $\overline{\partial}_n f_2 = \overline{L(f_1)}$ $\Rightarrow$ condition C_2 holds.

4.5 The case of the unit ball

For Theorem 6 we use a computation made by Kytmanov, who proved the analogous result for holomorphic extensions of homogeneous polynomials.

For Theorem 7, we suppose $f_1 \in \mathcal{H}_{p,q}$ and use Gauss formula for the harmonic extension into B of the trace $f_{1|S}$:

$$\tilde{f}_1 = \sum_{s \geq 0} g_{p-s,q-s},$$

where $g_{p-s,q-s}$ is the homogeneous harmonic polynomial of degree $p+q-2s$ defined by

$$g_{p-s,q-s} = c_{p,q,s} \sum_{j \geq 0} \frac{(-1)^j (p+q-j-2s)!}{j!} |z|^{2j} \Delta^{j+s} f_1.$$

The equation $\overline{\partial}_n f_2 = \overline{L(f_1)}$ can now be solved easily since

$$\overline{\partial}_n \overline{L(g_{p-s,q-s})} = (p-s+1)\overline{L(g_{p-s,q-s})}.$$

4.6 Bases of U_k^ψ

Let $\mathcal{B}_{p,q}$ denote a complex base of the space $\mathcal{H}_{p,q}(S)$ $(p+q=k)$. Then:

(i) if $k=2m$ is even, a basis of U_k^ψ over $\mathbb{H}$ is given by the set

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q=k, 0 \le q \le p \le k\}.$$

(ii) if $k=2m+1$ is odd, a basis of U_k^ψ over $\mathbb{H}$ is given by

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q=k, 0 \le q < p \le k\}$$

$$\cup \ \{R(h_1), \dots, R(h_{m+1})\},$$

where $h_1, \dots, h_{m+1}$ are chosen such that the set

$$\left\{h_1, \frac{1}{p+1}\overline{L(h_1)}, \dots, h_{m+1}, \frac{1}{p+1}\overline{L(h_{m+1})}\right\}$$

forms a complex basis of $\mathcal{H}_{m,m+1}(S)$.

4.7 The product in $M^\infty(B)$

On the unit ball we have explicit formulas for harmonic continuation of polynomials and for the operator R.

Example. *The product of the ψ-regular, not holomorphic function*

$$f = (\bar{z}_1 + \bar{z}_2) + (\bar{z}_2 - \bar{z}_1)j$$

with itself is the ψ-regular function

$$m_B(f, f) = (2\bar{z}_1^2 + 4z_1\bar{z}_2) + (4z_1\bar{z}_2 - 2\bar{z}_1^2)j$$

and the product of f and $g = z_1 - z_1 j$ is

$$m_B(f, g) = m_B(g, f) =$$
$$(|z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 + 1) + (|z_2|^2 - |z_1|^2 + \bar{z}_1\bar{z}_2 - 1)j.$$