Perspectives of algebraic surfaces with $p_g = 0$

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S smooth projective surface over $\mathbb{C}$;

- $p_g(S) := h^0(S, \Omega^2_S)$;
- $q(S) := \frac{1}{2} b_1(S) = h^0(S, \Omega^1_S)$.

120 years ago: M. Noether posed the following:

Question

Is a surface S with $p_g = q = 0$ rational?

Negative answer: Enriques (1895), Campedelli, Godeaux ('30ies).

Mumford, 1980 in Montreal: Can a computer classify surfaces with $p_g = 0$?

1. Let S be a minimal surface of general type. Then:
   - $K_S^2 \geq 1$;
   - $\chi(S) := 1 - q(S) + p_g(S) \geq 1$.

   In particular, $p_g = 0 \implies q = 0$.
2. If K_S is not ample, then there are rational curves C such that $K_S.C = 0$, $C^2 = -2$.
   Contracting these curves one gets the *canonical model* X, having R.D.P.s as singularieties (i.e., locally $\mathbb{C}^2/\Gamma$, where $\Gamma \leq SL(2, \mathbb{C})$).

Theorem (Bombieri, Gieseker)

For all $(x, y) \in \mathbb{N} \times \mathbb{N}$ there is a quasiprojective variety $\mathfrak{M}_{(x,y)}$, which is a coarse moduli space for canonical surfaces with $(\chi(X), K_X^2) = (x, y)$.

Theorem (Bogomolov-Miyaoka-Yau)

Let S be a surface of general type. Then:

- $K_S^2 \leq 9\chi(S)$;
- $K_S^2 = 9\chi(S)$ *iff the universal covering of* S *is the complex ball*
  - $\mathbb{B}_2 := \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 < 1\}$.

This means: we need to understand the nine moduli spaces

$$\mathfrak{M}_{(1,k)}, \; 1 \leq k \leq 9.$$

Remark

1. It is not easy to decide whether two surfaces are in the same connected component of the moduli space.
2. Easy observation: if S, S' are in the same connected component of $\mathfrak{M}$, they are orientedly diffeomorphic, hence homeomorphic. In particular, $\pi_1(S) = \pi_1(S')$.

Some open problems:

1. What are the π_1's of surfaces of general type with $p_g = 0$?
2. Is $\pi_1(S)$ residually finite for $p_g(S) = 0$?
3. What are the best possible numbers a, b such that
   - $K_S^2 \leq a \implies |\pi_1(S)| < \infty$,
   - $K_S^2 \geq b \implies |\pi_1(S)| = \infty$.

4. Are all surfaces with $p_g = 0$, $K_S^2 = 8$ uniformized by $\mathbb{H} \times \mathbb{H}$?
5. Conjecture (M. Reid): $\mathfrak{M}_{(1,1)}$ has exactly 5 irreducible components corresponding to $\pi_1(S) = \mathbb{Z}/m\mathbb{Z}$, $1 \leq m \leq 5$.
6. $K_S^2 = 2 \implies |\pi_1(S)| \leq 9$?
7. $K_S^2 = 3 \implies |\pi_1(S)| \leq 16$?

Conjecture (Bloch)

Let S be a smooth surface with $p_g(S) = 0$. Then

$$T(S) := \ker(A_0^0(S) \to \mathrm{Alb}(S)) = 0,$$

where $A_0^0(S)$ is the group of rational equivalence classes of zero cycles of degree zero.

Other reasons, why surfaces with $p_g = 0$ are interesting:

- pluricanonical maps, in particular the bicanonical map;
- differential topology: simply connected surfaces of general type with $p_g = 0$ are homeomorphic to Del Pezzo surfaces, but not diffeomorphic.

Ballquotients

$S = \mathbb{B}_2/\Gamma$, where $\Gamma \leq PSU(2,1)$ is a discrete, cocompact, torsionfree subgroup.

Remark

1) S is rigid. In particular, $\mathfrak{M}_{(1,9)}$ consists of isolated points.
2) Breakthrough 2003: Γ is arithmetic (Klingler).

Theorem (Prasad-Yeung)

$\mathfrak{M}_{(1,9)}$ consists of 100 isolated points, corresponding to 50 pairs of complex conjugate surfaces.

Product-quotient surfaces

We consider the following construction

- C_1, C_2 projective curves of resp. genera $g_1, g_2 \geq 2$;
- G finite group acting faithfully on C_1 and C_2;
- S a minimal model of a minimal resolution of singularities $S' \to X := (C_1 \times C_2)/G$.

Remark

1) The above surfaces are called PRODUCT-QUOTIENT SURFACES.
2) The geometry of these surfaces is encoded in certain algebraic data of G.
Therefore a systematic search of such surfaces can be carried through with a computer algebra program.

Product-quotient surfaces

Theorem (B., Catanese, Grunewald, Pignatelli)

1. *Surfaces* isogenous to a product, *i.e.*, $(C_1 \times C_2)/G$ *smooth, form17 irreducible connected components of* $\mathfrak{M}_{(1,8)}$.
2. *Surfaces such that* $X := (C_1 \times C_2)/G$ *have R.D.P.s form 27 irreducible families.*
3. S' *minimal and* X *does not have R.D.P.s form 32 families.*

Remark

By a result of S. Kimura, all the above surfaces satisfy Bloch's conjecture.

We can compute π_1 of all these surfaces. More precisely, we have the following structure theorem:

Theorem (-, Catanese, Grunewald, Pignatelli)

There is a normal subgroup $\mathcal{N}$ of finite index in $\pi_1(X)$, s. th. $\mathcal{N} \cong \pi_g \times \pi_{g'}$, for some $g, g' \geq 0$.

Coverings

Systematic (computer aided) search for surfaces with $p_g = 0$, which are abelian covers of e.g. $\mathbb{P}^2$ branched in line configurations.
Work in progress (S. Coughlan).

Theorem (Coughlan)

There is a surface S with $p_g = 0$, $K_S^2 = 8$ not uniformized by $\mathbb{H} \times \mathbb{H}$.

Large fundamental groups

Suppose $\pi_1(S)$ sits in an exact sequence

$$1 \to \pi_{g_1} \times \ldots \times \pi_{g_r} \to \pi_1(S) \to G' \to 1,$$

where $g_i \geq 1$, G' a finite group.

Question

What can one say about S?

Theorem (Catanese)

$S = (C_1 \times C_2)/G$, G finite group acting freely $\implies$

$$1 \to \pi_1(C_1) \times \pi_1(C_2) \to \pi_1(S) \to G \to 1,$$

and: if S' has the same π_1 and the same topological Euler characteristic as S, then S' or $\bar{S}'$ is in the same irreducible connected component as S.

Question

Suppose S' is homotopically equivalent to S. Under which conditions is S' in the same irreducible (connected) component as S?

We have

$$\begin{array}{ccc} \hat{S} & \longrightarrow & C_1 \times \ldots \times C_r \\ {\scriptstyle G'} \big\downarrow & & \\ S & & \end{array}$$

where $\pi_1(\hat{S}) = \pi_1(C_1) \times \ldots \times \pi_1(C_r)$.

E.g., $r = 2$: generically finite map $\hat{S} \to C_1 \times C_2$, study this map; $r = 3$: if e.g., $\hat{S} \hookrightarrow C_1 \times C_2 \times C_3$.

This has been carried through in the following cases:

1. Keum-Naie surfaces (B.-Catanese):

$$\begin{array}{ccc} \hat{S} & \longrightarrow & E_1 \times E_2 \qquad d.c. \\ {\scriptstyle (\mathbb{Z}/2\mathbb{Z})^2} \downarrow & & \\ S & & \end{array}$$

2. primary Burniat surfaces (B.-Catanese):

$$\begin{array}{ccc} \hat{S} & \subset & E_1 \times E_2 \times E_3 \qquad (2,2,2)\ h.s. \\ {\scriptstyle (\mathbb{Z}/2\mathbb{Z})^3} \downarrow & & \\ S & & \end{array}$$

3. Kulikov surfaces (Chan-Coughlan):

$$\begin{array}{ccc} \hat{S} & \subset & E_1 \times E_2 \times E_3 \qquad (3,3,3)\ h.s. \\ {\scriptstyle (\mathbb{Z}/3\mathbb{Z})^3} \downarrow & & \\ S & & \end{array}$$

Theorem

Each surface homotopically equivalent to one of the above surfaces is a surface as above. In particular, Keum-Naie, primary Burniat and Kulikov surfaces form an irreducible connected component of resp. dimension $6, 4, 1$ *in the moduli space.*

Burniat surfaces were constructed by P. Burniat in 1966 as singular bidouble covers of the projective plane.
$P_1, P_2, P_3 \in \mathbb{P}^2$,

- $D_1 := \{\delta_1 = 0\} = P_1 * P_2$ and two further lines containing P_1,
- $D_2 := \{\delta_2 = 0\} = P_2 * P_3$ and two further lines containing P_2,
- $D_3 := \{\delta_3 = 0\} = P_3 * P_1$ and two further lines containing P_3.

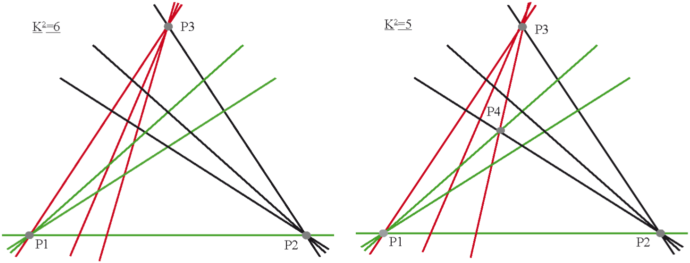
Definition

A minimal model S of a bidouble cover of $\mathbb{P}^2$ branched in (D_1, D_2, D_3) is called a Burniat surface.

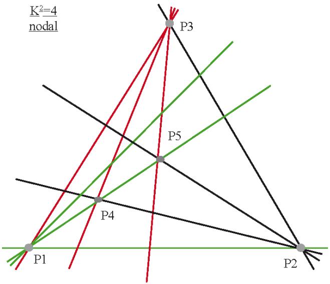
Remark

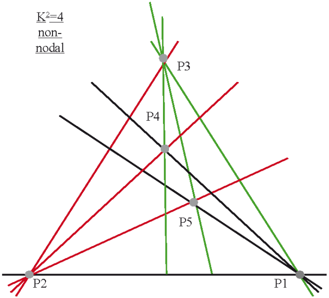
Burniat surfaces are surfaces of general type with $p_g(S) = q(S) = 0$ and $K_S^2 = 6 - m$, $1 \leq m \leq 4$.

Burniat configurations for $m = 0, 1$

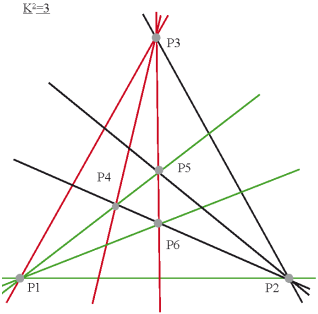


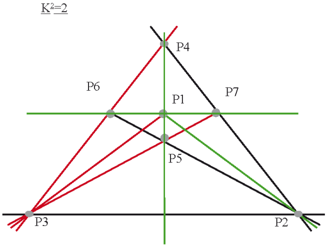
Burniat configurations for $m = 2$





Burniat configurations for $m = 3, 4$





We have the following results:

Theorem (B.-Catanese)

Burniat surfaces with $K_S^2 = 6, 5$ and Burniat surfaces with $K_S^2 = 4$ of non nodal type form a rational, irreducible connected component of the moduli space $\mathfrak{M}_{(1,K^2)}$.

Theorem (B.-Catanese)

Burniat surfaces with $K_S^2 = 4$ of nodal type (resp. with $K_S^2 = 3$) deform to extended nodal Burniat surfaces, *which form an irreducible connected (resp. irreducible) component of the moduli space $\mathfrak{M}_{(1,K^2)}$.*

Limits

- T = smooth affine curve, $0 \in T$, $f \colon \mathcal{X} \to T$ flat family of canonical surfaces and suppose that $\mathcal{X}_t$ is the canonical model of a Burniat surface with $K^2_{\mathcal{X}_t} \geq 4$, $t \neq 0$.
- Then there is an action of $G := (\mathbb{Z}/2\mathbb{Z})^2$ on $\mathcal{X}$ yielding a 1-parameter family of finite G-covers $\mathcal{X}_t \to \mathcal{Y}_t$, where $\mathcal{Y}_t$ is a Gorenstein Del Pezzo surface $\forall t$.
- The branch locus of $\mathcal{X}_0 \to \mathcal{Y}_0$ is the limit of the branch loci of $\mathcal{X}_t \to \mathcal{Y}_t$, hence
- $\mathcal{Y}_0$ cannot have worse singularities than $\mathcal{Y}_t$
- $\implies \mathcal{X}_0$ is again a Burniat surface.

Deformations of nodal Burniat surfaces

m=2:
$W := \hat{\mathbb{P}}^2(P_1, \dots, P_5)$ weak Del Pezzo surface,
$N := L - E_1 - E_4 - E_5$ nodal curve.
Extended nodal Burniat surface:
bidouble cover branched on $\Delta_1 + \Delta_2 + \Delta_3$ on W, where:

- $D_1 = (L - E_1 - E_2) + (L - E_1) + (L - E_1 - E_4 - E_5) + E_3$ (B);
- $\Delta_1 = D_1 - N = (L - E_1 - E_2) + (L - E_1) + E_3$ (extended B);
- $D_2 = (L - E_2 - E_3) + (L - E_2 - E_4) + (L - E_2 - E_5) + E_1$ (B);
- $\Delta_2 \equiv D_2 + N = (2L - E_2 - E_3 - E_4 - E_5) + (L - E_2 - E_4) + (L - E_2 - E_5)$ (extended B);
- $D_3 = (L - E_1 - E_3) + (L - E_3 - E_4) + (L - E_3 - E_5) + E_2$ (B);
- $\Delta_3 = D_3 + N$ (extended B).

Remark

Extended Burniat surfaces are bidouble covers of weak Del Pezzo surfaces, but the branch locus varies discontinuously.

S minimal model of a nodal Burniat surface with $K_S^2 = 4$, X its canonical model.

$$\begin{array}{ccc} S & \longrightarrow & X \\ \Big\downarrow{\scriptstyle (\mathbb{Z}/2\mathbb{Z})^2} & & \Big\downarrow{\scriptstyle (\mathbb{Z}/2\mathbb{Z})^2} \\ W := \hat{\mathbb{P}}^2(P_1, \ldots, P_5) & \longrightarrow & Y \subset \mathbb{P}^4 \end{array}$$

Consider:

- $\mathrm{Def}(S)$:= base of the Kuranishi family of S;
- $\mathrm{Def}(X)$:= base of the Kuranishi family of X;

Theorem (Burns-Wahl)

There is a fibre product

$$\begin{array}{ccc} \mathrm{Def}(S) & \longrightarrow & \mathcal{L}_S \cong \mathbb{C}^\nu \\ \downarrow & & \downarrow w \\ \mathrm{Def}(X) & \longrightarrow & \mathcal{L}_X \cong \mathbb{C}^\nu, \end{array}$$

where $\mathcal{L}_X$ is the space of local deformations of $\mathrm{Sing}(X)$, $\nu :=$ *number of (-2)-curves on S.*

Corollary

1) $\mathrm{Def}(S) \to \mathrm{Def}(X)$ *is finite;*
2) if $\mathrm{Def}(X) \to \mathcal{L}_X$ *is not surjective, then* $\mathrm{Def}(S)$ *is singular.*

Assume $G \leq \mathrm{Aut}(S) = \mathrm{Aut}(X)$, then
$\mathrm{Def}(S, G) = \mathrm{Def}(S)^G = \{J_t | g \in G \text{ is } J_t - \text{holomorphic}\}$.

Theorem (B.-Catanese)

The deformations of nodal Burniat surfaces to extended nodal Burniat surfaces exist and yield examples where $\mathrm{Def}(S, G) \to \mathit{Def}(X, G) = \mathit{Def}(X)$ *is not surjective.*

But each deformation of a nodal Burniat surface has a $G = (\mathbb{Z}/2\mathbb{Z})^2$-action.

The reason is local

$G = (\mathbb{Z}/2\mathbb{Z})^2 = \{1, \sigma_1, \sigma_2, \sigma_3 = \sigma_1 + \sigma_2\}$ acts on the family $\{X_t\}$,

$$X_t := \{w^2 = uv + t\},$$

by $\sigma_1(u, v, w) = (u, v, -w)$, $\sigma_2(u, v, w) = (-u, -v, w)$, with quotient

$$Y := \{z^2 = xy\}.$$

$\{X_t\}$ admits a simultaneous resolution only after the base change $\tau^2 = t$:

$$\mathcal{X} := \{w^2 - \tau^2 = uv\}.$$

2 small resolutions:

$$\mathcal{S} := \{((u,v,w,\tau),\xi) \in \mathcal{X} \times \mathbb{P}^1 : \frac{w-\tau}{u} = \frac{v}{w+\tau} = \xi\},$$

$$\mathcal{S}' := \{((u,v,w,\tau),\eta) \in \mathcal{X} \times \mathbb{P}^1 : \frac{w+\tau}{u} = \frac{v}{w-\tau} = \eta\}.$$

G has several liftings to $\mathcal{S}$, but

- either it acts not biregularly (only birationally),
- or it acts biregularly, but does not leave τ fixed.

E.g., σ_2 acts biregularly on $\mathcal{S}$, but σ_1, σ_3 act only birationally.