On Severi varieties of nodal curves on K3 surfaces

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K3 surfaces

In this talk I will present some irreducibility result concerning Severi varieties of nodal curves on K3 surfaces.

A *K*3 surface X is a smooth complex projective surface with $\Omega^2_X \cong \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 0$.

A primitive $K3$ surface of genus g is a pair (X, L), with X a $K3$ surface, and L an indivisible, ample line bundle on X, such that $L^2 = 2g - 2$.

Then $|L|$ is base point free. It is very ample if and only if it does not contain any hyperelliptic curve. In this case the map determined by $|L|$ is an isomorphism of X onto a smooth surface of degree $2g - 2$ in $\mathbb{P}^g$ whose smooth hyperplane sections are canonical curves of genus g.

The basic example $g = 3$ is a smooth quartic surface in $\mathbb{P}^3$.

For all $g \geq 2$, we can consider the moduli space (or stack) $\mathscr{B}_g$ of primitive $K3$ surfaces of genus g, which is smooth, of dimension 19.

For (X, L) very general in $\mathscr{B}_g$, the Picard group of X is generated by the class of L.

Severi varieties

Let (X, L) be a $K3$ surface of genus g and let k and h be integers. Consider

$$V_{k,h}(X, L) := \{C \in |kL| \text{ irreducible and nodal with } g(C) = h\},$$

where $g(C)$ is the geometric genus of C, so that C has $\delta = g - h$ nodes.

$V_{k,h}(X, L)$, called the (k, h)–Severi variety of (X, L), is a locally closed subscheme of the projective space

$$|kL| \cong \mathbb{P}^{p_a(k)} \quad \text{where} \quad p_a(k) := 1 + k^2(g - 1)$$

is the arithmetic genus of the curves in $|kL|$.

Warning

- I will drop the index k in $V_{k,h}(X, L)$ if $k = 1$, the only case I will actually consider!
- I may drop the indication of the pair (X, L) in $V_{k,h}(X, L)$ if there is no danger of confusion.

Dimension and degree of Severi varieties

Theorem (F. Severi (1921), S. T. Yau–E. Zaslov (1996), A. Beauville (1999), Xi Chen (1999))

Let $k \geq 1$ and $0 \leq h \leq p_a(k)$.

(i) *The variety $V_{k,h}$, if not empty, is smooth of dimension h.*

(ii) *If (X, L) is general, all irreducible, rational curves in $|L|$ are in V_0, which consists of $n(g)$ points, where*

$$\sum_g n(g)x^g = \frac{1}{\prod_k (1 - x^k)^{24}}$$

(iii) *If (X, L) is general in $\mathscr{B}_g$, then $V_{k,h}$ is not empty.*

Problem

Compute:

- the Hilbert polynomial of the closure of $V_{k,h}$ in $|kL|$;
- the arithmetic and geometric genera of a desingularization of $V_{k,h}$, at least if $1 \leq h \leq 2$.

An example

Consider the case $g = 3$, $k = 1$, i.e. look at Severi varieties V_h, $0 \leq h \leq 2$, of a general quartic surface $X \subset \mathbb{P}^3$:

- the closure of V_2 is the dual surface X^* of X, hence it has degree 36;
- the closure of V_1 is the nodal curve of X^*. Its degree, which can be computed using the formulas we learned from Y. Tzeng yesterday, is 480 and was classically computed by G. Salmon (1846);
- V_0 consists of $n(3)$ points, which are triple for both V_1 and V_2. According to Yau–Zaslov's formula

$$\boxed{n(3) = 3200}$$

This number was also classically computed by G. Salmon (1846).

Problem

Compute the geometric genus of V_1.

This probably requires computing the number of tacnodal plane sections of X.

The strong irreducibility conjecture

Conjecture (Strong irreducibility conjecture (SIC))

If (X, L) is general in $\mathscr{B}_g$ and $h \geq 1$, then $V_{k,h}$ is irreducible.

The SIC is trivially true for $h = p_a(k)$ and not difficult to prove for $h = p_a(k) - 1$, i.e. $\delta = 1$: in this case the closure of $V_{k,h}$ is the dual hypersurface X^* of X in $\mathbb{P}^{p_a(k)}$.

Proving SIG seems to be quite complicated as soon as $\delta \geq 2$, and, even for $\delta = 2$, it has never been established for all g and k.

Problem

If (X, L) is general in $\mathscr{B}_g$:

- describe in detail the singular locus of X^*;
- less ambitiously, is it true that it has only two irreducible components, corresponding to binodal and to cuspidal curves?

A weaker conjecture concerns universal Severi varieties.

For any $g \geq 2$, $k \geq 1$ and $0 \leq h \leq p_a(k)$, one can consider the functorially defined universal Severi variety (or stack) $\mathscr{V}^g_{k,h}$, which is pure and smooth of dimension $19 + h$, endowed with a morphism

$$\boxed{\phi^g_{k,h} : \mathscr{V}^g_{k,h} \longrightarrow \mathscr{B}^\circ_g}$$

which is smooth on all components of $\mathscr{V}^g_{k,h}$ and $\mathscr{B}^\circ_g$ is a suitable dense open substack of $\mathscr{B}_g$.

The fibres are described in the following diagram

$$\begin{array}{ccc} \mathscr{V}^g_{k,h} & \supset & V_{k,h}(X,L) \\ {\scriptstyle \phi^g_{k,h}}\downarrow & & \downarrow \\ \mathscr{B}^\circ_g & \ni & (X,L) \end{array}$$

i.e., a closed point of $\mathscr{V}^g_{k,h}$ can be regarded as a pair (X, C) with $(X, L) \in \mathscr{B}_g$ and $C \in V_{k,h}(X, L)$.

The weak irreducibility conjecture

Conjecture (Weak irreducibility conjecture (WIC), Th. Dedieu (2009))

$\mathscr{V}^g_{k,h}$ is irreducible for any $g \geq 2$, $k \geq 1$ and $0 \leq h \leq p_a(k)$.

Warning

- The SIC implies the WIC but not conversely.
- The WIC means that the monodromy of $\phi^g_{k,h}$ acts transitively on the components of the fibres $V_{k,h}(X, L)$, for $(X, L) \in \mathscr{B}_g$ general.
- The WIC makes sense even if $h = 0$, when $V_{k,0}(X, L)$ is certainly reducible.

The WIC was originally motivated by the attempt of proving the following result recently proved by Xi Chen:

Theorem (Xi Chen (2010))

If (X, L) is general in $\mathscr{B}_g$, there is no rational map $f : X \dashrightarrow X$ with $\deg(f) > 1$.

The WIC implies the theorem, but Chen proves it in a different way with degeneration techniques.

A result on the WIC

Theorem (C.-Th. Dedieu (2010))

For $3 \leq g \leq 11$, $g \neq 10$ and $0 \leq h \leq g$, the universal Severi variety $\mathscr{V}_h^g$ is irreducible.

Idea of the proof

We adopt a Hilbert schematic viewpoint inspired by a paper of C.–A. Lopez–R. Miranda (1998).

We find a certain flag Hilbert scheme $\mathscr{F}_{g,h}$, with a rational map

$$\mathscr{F}_{g,h} \dashrightarrow \mathscr{V}_h^g$$

dominating all components of $\mathscr{V}_h^g$, and we prove that $\mathscr{F}_{g,h}$ is irreducible.

To show the irreducibility of $\mathscr{F}_{g,h}$, we exhibit smooth points of $\mathscr{F}_{g,h}$ which must be contained in all irreducible components of $\mathscr{F}_{g,h}$.

Next I will provide some details.

The moduli map

There is a natural moduli map

$$\mu^g_{k,h} : \mathscr{V}^g_{k,h} \to \mathscr{M}_h$$

where $\mathscr{M}_h$ is the moduli space of curves of genus h.

The case $k = 1$, $h = g$ has been much studied in the past. Only recently the nodal case $h < g$, $k = 1$, received the deserved attention.

Theorem

Assume $3 \leq g \leq 11$ and $0 \leq h \leq g$. For any irreducible component $\mathscr{V}$ of $\mathscr{V}^g_h$, the moduli map $\mu^g_{h|\mathscr{V}} : \mathscr{V} \to \mathscr{M}_h$ is dominant, unless $g = h = 10$.

The case $h = g$ is due to S. Mori–S. Mukai (1983) and to S. Mukai (1988/1992). The nodal case is in a recent paper by F. Flamini–A. L. Knutsen–G. Pacienza–E. Sernesi (2008).

Recently M. Halic (2009) studied $\mu^g_{k,h}$ also for $g \geq 13$, $k \geq 2$ and h sufficiently large with respect to g, proving that, as expected, $\mu^g_{k,h}$ is generically finite to its image in these cases.

The remaining cases $k \geq 2$ and h low with respect to g are still unexplored.

The Hilbert scheme viewpoint (I)

For any $g \geq 3$, we let $\mathcal{B}_g$ be the component of the Hilbert scheme of surfaces in $\mathbb{P}^g$ parametrizing primitive $K3$ surfaces of genus g. This is generically smooth of dimension

$$\boxed{\dim(\mathcal{B}_g) = g^2 + 2g + 19}$$

Let $\mathcal{C}_g$ be the component of the Hilbert scheme of curves in $\mathbb{P}^g$ parametrizing degenerate canonical curves of genus g, i.e. smooth canonical curves of genus g lying in a hyperplane of $\mathbb{P}^g$. This is generically smooth of dimension

$$\boxed{\dim(\mathcal{C}_g) = g^2 + 4g - 4}$$

Let $\mathcal{F}_g$ be the component of the flag Hilbert scheme of $\mathbb{P}^g$ whose general point is a pair (X, C) with $X \in \mathcal{B}_g$ general and $C \in \mathcal{C}_g$ a general hyperplane section of X. This is generically smooth of dimension

$$\boxed{\dim(\mathcal{F}_g) = g^2 + 3g + 19}$$

The Hilbert scheme viewpoint (II)

Let $\mathcal{C}_{g,h}$ be the Zariski closure of the locally closed subset of $\mathcal{C}_g$ formed by irreducible, nodal curves with $\delta = g - h$. This is generically smooth, irreducible of dimension

$$\boxed{\dim(\mathcal{C}_{g,h}) = \dim(\mathcal{C}_g) - \delta = g^2 + 4g - 4 - \delta}$$

If $h \geq 0$, it comes with a dominant moduli map

$$c_{g,h} : \mathcal{C}_{g,h} \dashrightarrow \mathscr{M}_h.$$

We let $\mathcal{F}_{g,h}$ be the inverse image of $\mathcal{C}_{g,h}$ under the projection $\mathcal{F}_g \to \mathcal{C}_g$. This is the Hilbert schematic version of the universal Severi variety $\mathscr{V}_h^g$. Indeed there is a componentwise dominant, functorial map

$$m_{g,h} : \mathcal{F}_{g,h} \dashrightarrow \mathscr{V}_h^g$$

Hence it suffices for us to prove the following:

Theorem

Let $3 \leq g \leq 11$, $g \neq 10$, and $0 \leq h \leq g$. Then $\mathcal{F}_{g,h}$ is irreducible.

Any irreducible component $\mathcal{F}$ of $\mathcal{F}_{g,h}$ dominates $\mathcal{B}_g$ via the restriction of the projection $\mathcal{F}_g \to \mathcal{B}_g$ (see FKPS) and has dimension

$$\boxed{\dim(\mathcal{F}) = \dim(\mathcal{F}_{g,h}) = g^2 + 3g + 19 - \delta}$$

Let

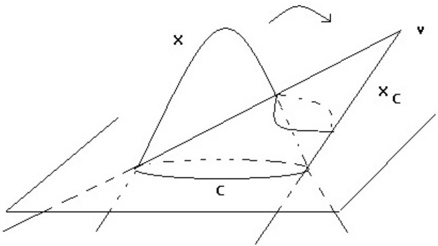
$$p_{g,h} : \mathcal{F}_{g,h} \longrightarrow \mathcal{C}_{g,h}$$

be the natural projection; I set p_g for $p_{g,g}$.

The following lemma relies on a construction by H. Pinkham (1974) and on the fact that smooth $K3$ surfaces are projectively Cohen–Macaulay.

Proposition

Let $(X, C) \in \mathcal{F}_{g,h}$ with X smooth. Let X_C be the cone over C with vertex a point v in $\mathbb{P}^g$ off the hyperplane in which C sits. Then one can flatly degenerate (X, C) to (X_C, C) inside the fibre F_C of $p_{g,h}$ over C.



Tangent spaces computations

The fibre F_C of

$$p_{g,h} : \mathcal{F}_{g,h} \to \mathcal{C}_{g,h}$$

over $C \in \mathcal{C}_{g,h}$ equals the fibre of p_g, and therefore

$$\boxed{T_{(X,C)}(F_C) \cong H^0(X, N_{X/\mathbb{P}^g}(-1))}$$

Next proposition computes this space at a cone point (X_C, C) and relies on the fact that C is projectively Cohen–Macaulay.

Proposition (H. Pinkham (1974))

Let C be a reduced and irreducible degenerate (not necessarily smooth), genus g canonical curve in $\mathbb{P}^g$, and let X_C be the cone over C with vertex a point in $\mathbb{P}^g$ off the hyperplane of C. For all $i \geq 0$, one has

$$\boxed{H^0(X_C, N_{X_C/\mathbb{P}^g}(-i)) \cong \bigoplus_{k \geq i} H^0(C, N_{C/\mathbb{P}^{g-1}}(-k))}$$

Next we bound from above the dimension of the cohomology spaces in the right–hand–side of the above formula: we use semi–continuity and a special type of canonical curves for which they can be computed.

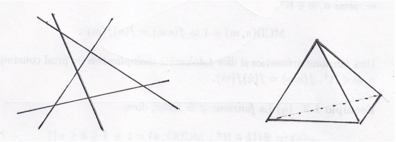
Canonical graph curves (I)

A graph curve of genus g is a stable curve of genus g consisting of $2g-2$ irreducible components of genus 0.

It has $3g-3$ nodes (three nodes for each component), and it is determined by the dual trivalent graph, consisting of $2g-2$ nodes and $3g-3$ edges.

If C is a graph curve and its dualizing sheaf ω_C is very ample, then C can be canonically embedded in $\mathbb{P}^{g-1}$ as a union of $2g-2$ lines, each meeting three others at distinct points. This is a canonical graph curve, and it is projectively Cohen–Macaulay (D. Bayer– D. Eisenbud, 1991).

The simplest example of a canonical graph curve is the union of 4 independent lines in the plane, whose dual graph is a tetrahedron:



Canonical graph curves (II)

Next proposition is based on Wahl maps computations for graph curves:

Proposition (C.–A. Lopez–R. Miranda (1998))

For any $3 \leq g \leq 11$, $g \neq 10$, there exists a genus g canonical graph curve Γ_g in $\mathbb{P}^{g-1}$, sitting in the image of p_g, such that the dimensions of the spaces of sections of non–positive twists of their normal bundles are given in the following table

$h^0\left(N_{\Gamma_g/\mathbb{P}^{g-1}}(-k)\right) \setminus g$	3	4	5	6	7	8	9	11
$k=0$	g^2+3g-4 *for every* g							
$k=1$	10	13	15	16	16	15	14	12
$k=2$	6	5	3	1	0	0	0	0
$k=3$	3	1	0	0	0	0	0	0
$k=4$	1	0	0	0	0	0	0	0
$k \geq 5$	0 *for every* g							

hence

$$\sum_{k \geq 1} h^0\left(\Gamma_g, N_{\Gamma_g/\mathbb{P}^{g-1}}(-k)\right) = 23 - g$$

for these curves.

The proof of the theorem: first step

Proposition

Let g and h be two integers such that $3 \leq g \leq 11$, $g \neq 10$, and $0 \leq h \leq g$. Let $\mathcal{F}$ be a component of $\mathcal{F}_{g,h}$ and let $(X, C) \in \mathcal{F}$ be general. All components of $F_C = p_{g,h}^{-1}(p_{g,h}(X, C))$ have dimension $23 - g$, and the restriction of $p_{g,h}$ to $\mathcal{F}$ dominates $\mathcal{C}_{g,h}$.

Proof

Recall that $X \in \mathcal{B}_g$ is general. Moreover F_C equals the fibre of p_g. Thus

$$\dim(T_{(X,C)}(F_C)) \cong h^0(X, N_{X/\mathbb{P}^g}(-1)) = \dim(T_{(X,\bar{C})}(F_{\bar{C}}))$$

with $\bar{C}$ a general hyperplane section of X, so that $\bar{C} \in \mathcal{C}_g$ is general.

By first degenerating to the cone point $(X_{\bar{C}}, \bar{C})$, and then to one of the graph curves in the CLM Proposition, we have

$$\dim_{(X,C)}(F_C) \leq h^0(X, N_{X/\mathbb{P}^g}(-1)) \leq 23 - g$$

Since

$$23 - g = \dim(\mathcal{F}) - \dim(\mathcal{C}_{g,h})$$

this equals the dimension of F_C at (X, C), and the restriction of $p_{g,h}$ at $\mathcal{F}$ is dominant.

The proof of the theorem: conclusion

With a similar argument we finish the proof of the theorem, showing that $\mathcal{F}_{g,h}$ is irreducible.

Proof

Let $\mathcal{F}_i$, $1 \leq i \leq 2$, be distinct components of $\mathcal{F}_{g,h}$. Let $C \in \mathcal{C}_{g,h}$ be a general point.

By the Proposition in the previous slide, there are points $(X_i, C) \in \mathcal{F}_i$, and they can be assumed to be general points on two distinct components F_i of F_C, $1 \leq i \leq 2$.

By the degenerations to cone Proposition, F_1 and F_2 both contain the cone point (X_C, C).

Since C is general in $\mathcal{C}_{g,h}$ and $\mathcal{C}_{g,h}$ contains the graph curves Γ_g of the CLM Proposition (because $\mathcal{C}_{g,h}$ contains $\mathcal{C}_{g,h-1}$), by upper–semicontinuity $h^0(X_C, N_{X_C/\mathbb{P}^g}(-1))$ is bounded above by $23 - g$.

This proves that F_C is smooth at (X_C, C), giving a contradiction, which concludes the proof.

Two remarks

(i) The above argument gives an alternative quick proof of the part $h < g$ of the FKPS Theorem when $g \neq 10$.

(ii) By contrast, the argument of the proof does not work for $g = 10$ (and even less for $g > 11$).

If C is any curve in the image of p_{10} lying on a smooth $K3$ surface, one has (Cukierman–Ulmer, 1993)

$$h^0\left(X, N_{X/\mathbb{P}^g}(-1)\right) = 14 = 23 - g + 1$$

Then all components of a general fibre of $p_{10,h}$ have dimension 14, which implies that the image of $p_{g,h}$ has codimension 1 in $\mathcal{C}_{g,h}$. The FKPS Theorem ensures that, as should be expected, $\mathcal{C}_{g,h}$ dominates $\mathscr{M}_h$ for $0 \leq h \leq 9$.

However the argument in the final part of the proof of our Theorem falls short, since we do not know whether the image of $p_{10,h}$ is irreducible, or all of its components contain a curve C for which the fibre F_C can be controlled.

Degenerations of Severi varieties

Next I propose a different, perhaps more direct, attack to the problem, which goes back to Z. Ran (1986). This is still work in progress. One can see the next slides as a set of footnotes to Y. Tzeng's talk.

Question

Suppose we have a flat, (simple) normal crossings degeneration, parametrised by a disc, of general polarised $K3$ surfaces (X, L) of genus g. How does the Severi variety $V_{k,h}(X, L)$ degenerate?

This has been treated by Xi Chen for the case $k = 1$, $h = 0$, i.e. he has been looking at the limit of g–nodal rational curves in $|L|$ for some degeneration.

I want to discuss in some detail the example $g = 3$, $k = 1$, $h = 0$, i.e. tritangent plane sections of a general quartic surface in $\mathbb{P}^3$.

I will indicate an hopefully instructive direct argument for the irreducibility of $\mathscr{V}_0^3$, which may shed some light on the structure of the monodromy of the map $\phi_0^3 : \mathscr{V}_0^3 \longrightarrow \mathscr{B}_3^\circ$.

I hope that these ideas can be extended to higher values of g, k, h.

Normal crossing degenerations

Suppose we have a simple global normal crossing degeneration (SGNCD) of a smooth, irreducible, projective surface, parametrized by a disc Δ, i.e. a proper and flat morphism

$$\begin{array}{c}\mathcal{X}\\ \downarrow{\scriptstyle \pi}\\ \Delta\end{array}$$

with $\mathcal{X}$ smooth, $X_t = \pi^{-1}(t)$ a smooth, irreducible, projective surface, for $t \neq 0$, and the central fibre

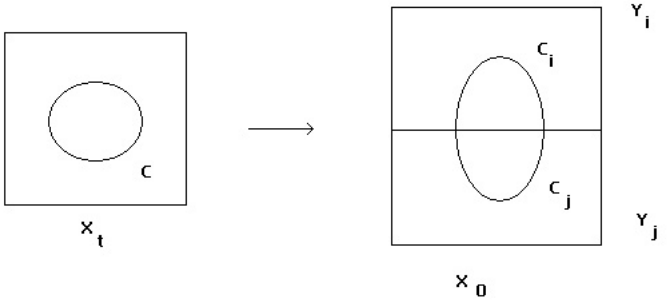
$$X_0 = \cup_i Y_i$$

with simple normal crossing and smooth components Y_i.

Assume there is a line bundle $\mathcal{L}$ on $\mathcal{X}$, which restricts to X_t as L_t. Then (X_t, L_t) degenerates to (X_0, L_0).

Limits of curves (I)

A curve C in $|L_t|$ degenerates, in general, to a union of curves C_i on the components Y_i of X_0 which match on the double curve of X_0.



Limits of curves (II)

Warning

It is not true in general that $C_i \in |L_{0|Y_i}|$, because the limit of the section corresponding to C may identically vanish on some component of X_0.

However if we assume $\mathcal{L}$ is tame, i.e. that:

- L_0 is centrally effective, i.e. its general section is non–zero on any component of the central fibre;
- $h^0(X_t, L_t)$ is constant in t;

then the general curve in $|L_0|$ is the limit of the general curve in $|L_t|$.

Warning

In general not all sections of L_0 correspond to curve, because special sections may vanish on some components of X_0.

Limits of linear systems inside the relative Hilbert scheme

Warning

By a previous warning, even in the tame case, can we cannot expect in general to find in $|L_0|$ all limits of curves in $|L_t|$.

This is because the bundle $\mathcal{L}$ may be not unique.

This may happen if X_0 is reducible, since we may then replace $\mathcal{L}$ by a twist

$$\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(\sum_i a_i Y_i)$$

which is still tame. This will change L_0 but not L_t for $t \neq 0$.

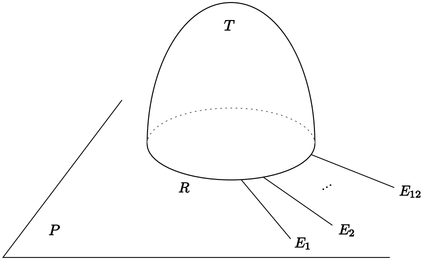
In this case the curves in the union of all limit linear systems $|L_0'|$ corresponding to all tame bundles $\mathcal{L}'$ as above appear as limits of curves in $|L_t|$.

This may be understood in terms of the relative Hilbert scheme stack construction indicated by Tseng yesterday.

An example (I)

We can make a degeneration (in a pencil) $\mathcal{X} \to \Delta$, of a general quartic surface X to a general union of a cubic surface T plus a plane Π, i.e. $X_0 = \Pi \cup T$. This is not a SGNCD: $\mathcal{X}$ is singular at the points corresponding to the 12 base points p_i of the pencil on the cubic curve $R = \Pi \cap T$.

To make it a SGNCD, one can make a small resolution of the singularities. The resulting new central fibre is then $X_0 = P \cup T$, where P is the blow–up of Π at the 12 points p_i, with exceptional curves E_i.



I will denote by $\mathcal{L}$ the pull back of the hyperplane bundle of $\mathbb{P}^3$.

An example (II)

There are exactly two tame bundles here, i.e.

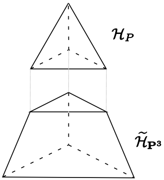
$$\mathcal{L} \quad \text{and} \quad \mathcal{L}' = \mathcal{L}(-P)$$

There is (up to a scalar) only one section s of $H^0(X_0, L_0)$ which does not correspond to a curve on X_0, i.e. the unique section vanishing on P.

$|L'_{0|P}|$ is the pull–back on P of the linear system of quartics on Π through the 12 points p_i, whereas $|L'_{0|T}|$ is trivial. So all sections in $H^0(X_0, L'_0)$ vanishing on R (and therefore on T) also do not correspond to curves on X_0.

The limit of $|L_t|$ for $t \neq 0$ in the relative Hilbert scheme $\mathcal{H}$ breaks up as follows:

$\mathcal{H}_P = |L'_{0|P}|$



$\tilde{\mathcal{H}}_{\mathbf{P3}} = \mathrm{Bl}_{(s=0)}(|L_0|)$

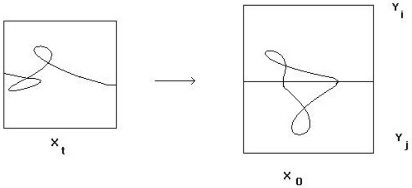
They are (suitably) glued along the exceptional divisor of $\tilde{\mathcal{H}}_{\mathbf{P^3}}$.

$\mathcal{H}$ is the blow–up of $\mathbb{P}(\pi_*(\mathcal{L}))$ at a point in the central fibre or the blow–up of $\mathbb{P}(\pi_*(\mathcal{L}'))$ over Δ along a plane in the central fibre.

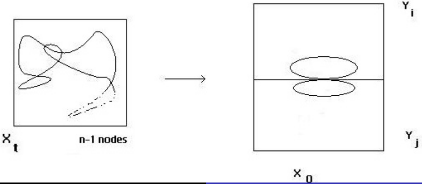
Limits of nodal curves

Limits of nodal curves in SGNCD's have been studied by Z. Ran (1986) (see also Tseng's talk).

Obvious limits of δ–nodal curves in $|L_t|$ are curves in a $|L_0|$ which have a total of δ nodes internal to the components of X_0:



Less obvious limits are the one indicated in the picture below: the two branches in the limit have both a contact of order n with the double curve. This is the limit of $n-1$ nodes and has to be counted with multiplicity n.



Limits of trinodal curves in the previous example

We have to compute trinodal curves in $\mathcal{H}$.

- in $\mathcal{H}_P$ we have 620 trinodal curves: this is also the degree of the Severi variety of plane trinodal quartics;
- in $\tilde{\mathcal{H}}_{\mathbf{P}^3}$ we have a total of 2580 trinodal curves (counted with multiplicities). The computation is shown in the table below:

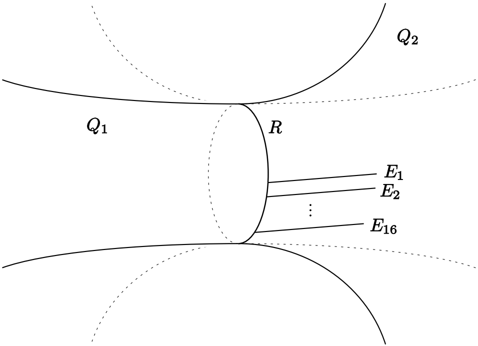
Type	Number
PXY	0
T^3	45
T^2E_i	324
T^2R	216
TE_iE_j	792
TRE_i	960
TR^2	0
μ_3RT	243
$E_iE_jE_k$	0
RE_iE_j	0
R^2E_i	0
μ_3RE_i	0
μ_4R	0
Total	2580

The total is 3200 as needed.

Degeneration of a quartic to a union of two quadrics

We can also make a degeneration (in a pencil) $\mathcal{X} \to \Delta$, of a general quartic surface X to a general union of two quadrics $X_0 = F_1 \cup F_2$. This is not a SGNCD: $\mathcal{X}$ is singular at the points corresponding to the 16 base points p_i of the pencil on the double curve R of X_0.

To make it a SGNCD, one can make a small resolution of the singularities. The resulting new central fibre is then $X_0 = Q_1 \cup Q_2$, where $Q_1 = F_1$ and Q_2 is the blow–up of F_2 at the 16 points p_i, with exceptional curves E_i.



There is only one tame bundle here.

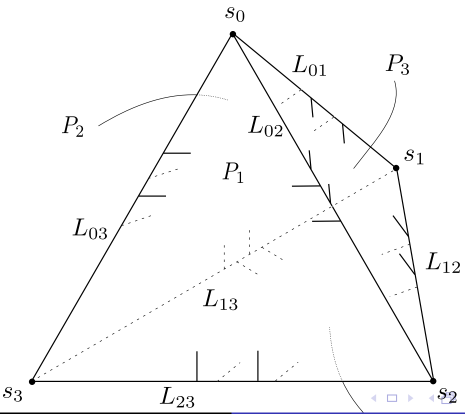
We can compute the limits of trinodal plane sections, which are given, with their multiplicities, in the table below:

Type	Number
$Q_i^2 X$	0
$Q_1 Q_2 E_i$	64
$Q_1 Q_2 R$	0
$Q_i E_r E_s$	480
$Q_i E_r R$	512
$Q_i R^2$	128
$Q_i \mu_3$R	0
R^3	0
μ_4R	64
$E_i R^2$	0
$E_i \mu_3 R$	432
$E_i E_j R$	960
$E_i E_j E_k$	560
Total	3200

Degeneration of a quartic to a union of four planes

Finally we make a degeneration (in a pencil) $\mathcal{X} \to \Delta$, of a general quartic surface X to a general union of four planes of a tetrahedron. Again this is not a SGNCD because $\mathcal{X}$ is singular at the 24 points corresponding to the intersections of the edges of the tetrahedron with the general surface of the pencil.

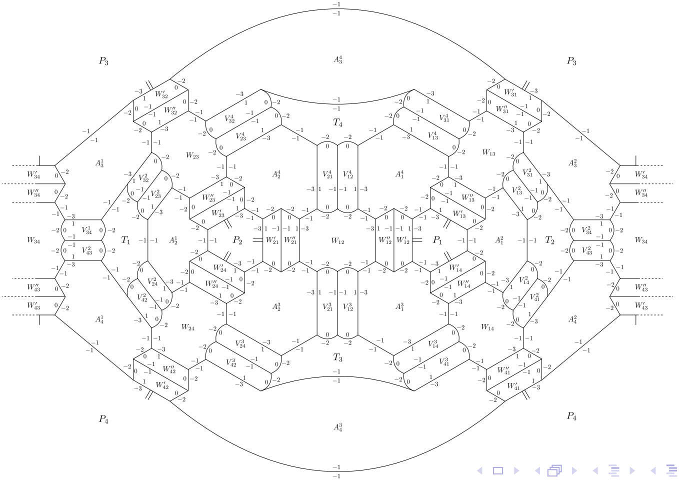
By making a small resolution of the singularities, we find the new central fibre which looks like:



Better model for the central fibre

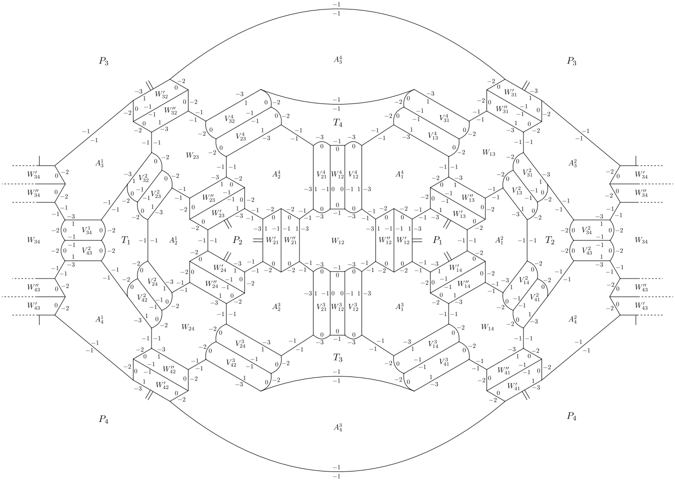
To make this amenable for computing the limit of trinodal plane sections, we resolve the vertices and the edges of the tetrahedron.

This is made by first performing an order 6 base change, then resolving the singularities of the resulting total space. The central fibre then looks like:



The best model for the central fibre

This is still not good enough. To make a good computation one has to blow up the double curves $V_{21}^4 \cap V_{12}^4$ and $V_{21}^3 \cap V_{12}^3$, producing two double components W_{12}^4 and W_{12}^3 of the central fibre:



Limits of trinodal plane sections

Now we compute the limits of trinodal plane sections (this computation has been done also by G. Mikhalkin and E. Brugallé using tropical geometry):

- planes through 3 blown up points on three pairwise skew edges: there are 1024 choices and each has multiplicity 1;
- planes through 2 blown up points on two distinct edges and through a vertex off both these edges: there are 192 choices, and each contributes 3 trinodal curves (this depends on the geometry of the components T_i), for a total of 576;
- planes through 1 blown up point and two vertices neither one lying on the edge of the point: there are 24 choices, and each contributes 16 trinodal curves (this depends on the geometry of the surfaces W_{ij}), for a total of 384;
- the four faces of the tetrahedron: this is like counting the number of trinodal curves in the linear system of quartic plane curves passing through 12 points 4 by 4 located along three independent lines. This number (again computed by degeneration) is 304 (for general points is instead 620), for a total of 1216.

Summing up one gets the required number 3200.

The monodromy

Comparing the three degenerations one sees that the monodromy of

$$\phi_0^3 : \mathscr{V}_0^3 \longrightarrow \mathscr{B}_3^\circ$$

is transitive on the general fibre, thus giving an alternative proof of the irreducibility of the universal Severi variety $\mathscr{V}_0^3$.

Problem

Use the above technique (and may be other degenerations) to understand what is the monodromy of $\phi_0^3 : \mathscr{V}_0^3 \longrightarrow \mathscr{B}_3^\circ$. Is it the full symmetric group?

E.g., we have computed the number 3200 also by degenerating a general quartic to a Kummer surface.

Problem

Use the degenerations of $K3$ surfaces of CLM to extend this argument and prove that the universal Severi variety $\mathscr{V}_0^g$ is irreducible for all g.