The birational geometry of moduli spaces of even spin curves

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September 8, 2010

A spin curve of genus g is a pair (C, η), with $[C] \in \mathcal{M}_g$ and $\eta \in \mathrm{Pic}^{g-1}(C)$ with $\eta^{\otimes 2} = K_C$ a *theta characteristic*.

$$\mathcal{S}_g := \{[C, \eta]\} \text{ moduli space of spin curves of genus } g.$$

There is an étale covering $\pi : \mathcal{S}_g \to \mathcal{M}_g, \quad \pi([C, \eta]) := [C]$.

Spin curves come in 2 types, odd and even: $\mathcal{S}_g = \mathcal{S}_g^- \coprod \mathcal{S}_g^+$,

$$\mathcal{S}_g^+ := \{[C, \eta] \in \mathcal{S}_g : h^0(C, \eta) \equiv 0 \text{ mod } 2\}$$

and

$$\mathcal{S}_g^- =:= \{[C, \eta] \in \mathcal{S}_g : h^0(C, \eta) \equiv 1 \text{ mod } 2\}.$$

$$\deg(\mathcal{S}_g^+/\mathcal{M}_g) = 2^{g-1}(2^g + 1) \quad \deg(\mathcal{S}_g^-/\mathcal{M}_g) = 2^{g-1}(2^g - 1).$$

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What sort of varieties are $\mathcal{S}_g^-$ and $\mathcal{S}_g^+$?

Theorem

(Farkas, Verra 2010)

1. *The compactified moduli space $\overline{\mathcal{S}}_g^-$ of odd spin curves is of general type for $g \geq 12$.*
2. *$\overline{\mathcal{S}}_g^-$ is uniruled for $g \leq 11$ (unirational for $g \leq 9$).*

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1. *$\overline{\mathcal{S}}_g^+$ is uniruled for $g < 8$ (parametrization via Nikulin surfaces).*
2. *$\kappa(\overline{\mathcal{S}}_8^+) = 0$; The Mukai model of $\overline{\mathcal{S}}_8^+$ is Calabi-Yau of dimension 21.*

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Remark

1. For $8 \leq g \leq 11$: $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{S}}_g^+$ have different Kodaira dimension!
2. $\kappa(\overline{\mathcal{M}}_g)$ unknown for $17 \leq g \leq 21$; $\kappa(\mathcal{A}_6)$ unknown.

$$\begin{array}{ccc} \mathcal{S}_g & \longrightarrow & \overline{\mathcal{S}}_g \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{M}_g & \longrightarrow & \overline{\mathcal{M}}_g \end{array}$$

Requirements for a compactification of $\mathcal{S}_g$:

- $\overline{\mathcal{S}}_g$ should be modular (i.e. represent a DM stack), have good singularities.
- $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ should be a finite branched covering.

Solution: Cornalba compactification using stable spin curves.

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A compactification of $\mathcal{S}_g$

Definition

A *stable* spin curve of genus g is a triple (X, η, β), where:

- X is a quasi-stable with $p_a(X) = g$.
- $\eta \in \mathrm{Pic}^{g-1}(X)$ is a line bundle such that $\eta_E = \mathcal{O}_E(1)$, for every rational component $E \subset X$ with $|E \cap \overline{X - E}| = 2$.
- $\beta : \eta^{\otimes 2} \to \omega_X$ is a sheaf-homomorphism such that $\beta_Z \neq 0$, for every non-exceptional component $Z \subset X$.

$\overline{\mathcal{S}}_g$ is the coarse moduli space associated to the stack of spin curves. There is a ramified map $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ given by $\pi([X, \eta, \beta]) = [\mathrm{st}(X)]$.

Example

If $[C_{xy} := C/x \sim y] \in \Delta_0 \subset \overline{\mathcal{M}}_g$, with $[C, x, y] \in \mathcal{M}_{g-1,2}$, describe points $[X, \eta, \beta] \in \pi^{-1}([C_{xy}])$:

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Two types of spin curves over C_{xy}:

- Spin curves corresponding to locally free sheaves on $X = C_{xy}$:

$$[C_{xy},\ \eta_C \in \mathrm{Pic}^{g-1}(C),\ \eta_C^{\otimes 2} = K_C(x+y)] \in \overline{\mathcal{S}}_g^+.$$

- Those corresponding to torsion free sheaves on C_{xy}; "blow-up" the node, get $X := C \cup_{x,y} E$, where $E \cong \mathbf{P}^1$.

$$[C \cup_{x,y} E,\ \ \eta_E = \mathcal{O}_E(1),\ \eta_C^{\otimes 2} = K_C] \in \overline{\mathcal{S}}_g^+.$$

Denote the closure in $\overline{\mathcal{S}}_g^+$ of these loci by A_0 and B_0 respectively. Set

$$\alpha_0 := [A_0],\ \beta_0 := [B_0] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+).$$

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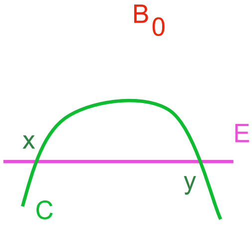
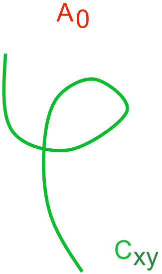
To summarize:

1. $\pi^*(\delta_0) = \alpha_0 + 2\beta_0$.
2. B_0 is the ramification divisor of π.





The canonical class of $K_{\overline{\mathcal{S}}_g^+}$

The Hurwitz formula applied to the branched covering $\pi : \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$:

$$K_{\overline{\mathcal{S}}_g^+} = \pi^*(K_{\overline{\mathcal{M}}_g}) + \beta_0 \equiv 13\lambda - 2\alpha_0 - 3\beta_0 - \cdots \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+).$$

Since singularities of $\overline{\mathcal{S}}_g^+$ impose no adjoint conditions (K. Ludwig), $\overline{\mathcal{S}}_g^+$ is of general type precisely when $K_{\overline{\mathcal{S}}_g^+}$ is big. To produce pluricanonical forms, we construct effective divisors $\mathfrak{D} \in \mathrm{Eff}(\overline{\mathcal{S}}_g^+)$, such that

$$K_{\overline{\mathcal{S}}_g^+} = a\lambda + b[\mathfrak{D}] + \mathbb{Q}_{\geq 0} \cdot (\text{boundary divisors}),$$

where $a > 0$ and $b \geq 0$.

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Effective divisors on $\overline{\mathcal{S}}_g^+$

The theta-null divisor

$$\Theta_{\mathrm{null}} := \{[C, \eta] \in \mathcal{S}_g^+ : h^0(C, \eta) \geq 2\}$$

$[C, \eta] \in \Theta_{\mathrm{null}} \Leftrightarrow \exists Q \in H^0(\mathcal{I}_{C/\mathbf{P}^{g-1}}(2))$ with $\mathrm{rk}(Q) = 3$, $C \cap \mathrm{Sing}(Q) = \emptyset$.

Theorem

The class of the closure of Θ_{null} inside $\overline{\mathcal{S}}_g^+$ equals:

$$\overline{\Theta}_{\mathrm{null}} \equiv \frac{\lambda}{4} - \frac{\alpha_0}{16} - \sum_{i=1}^{[g/2]} \frac{\beta_i}{2} \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+).$$

Remark

$\overline{\Theta}_{\mathrm{null}}$ has very small slope (good!) but coefficient of β_0 is 0 (bad!). Thus $\overline{\Theta}_{\mathrm{null}}$ alone will not suffice, to conclude that $K_{\overline{\mathcal{S}}_g^+}$ is big.

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Brill-Noether divisors

Fix integers $r, d \geq 1$ such that $\rho(g, r, d) = -1$. Set

$$\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : C \text{ has a } \mathfrak{g}^r_d\}.$$

Theorem

When $\rho(g, r, d) = -1$, the locus $\mathcal{M}^r_{g,d}$ is an irreducible divisor in $\mathcal{M}_g$. Moreover, the class of its closure in $\overline{\mathcal{M}}_g$ is:

$$\overline{\mathcal{M}}^r_{g,d} = c_{g,r,d}\Big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\Big).$$

Form a linear combination on $\overline{\mathcal{S}}^+_g$: $\quad (K_{\overline{\mathcal{S}}^+_g} \equiv 13\lambda - 2\alpha_0 - 3\beta_0 - \cdots)$

$$a \cdot \pi^*(\overline{\mathcal{M}}^r_{g,d}) + 8 \cdot \overline{\Theta}_{\mathrm{null}} \equiv \frac{11g+29}{g+1}\lambda - 2\alpha_0 - 3\beta_0 - \cdots .$$

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$\overline{\mathcal{S}}_g^+$ is a variety of general type whenever

$$\frac{11g+29}{g+1} < 13 \Longleftrightarrow g > 8.$$

For $g = 8$, the argument above shows that $\kappa(\overline{\mathcal{S}}_8^+) \geq 0$; $\exists a, a_i, b_i > 0$,

$$K_{\overline{\mathcal{S}}_8^+} \equiv a \cdot \pi^*(\overline{\mathcal{M}}_{8,7}^2) + 8 \cdot \overline{\Theta}_{\mathrm{null}} + \sum_{i=1}^{4} \big(a_i \cdot \alpha_i + b_i \cdot \beta_i\big),$$

where $\overline{\mathcal{M}}_{8,7}^2 \subset \overline{\mathcal{M}}_8$ is the irreducible locus of plane septics, and $\alpha_i, \beta_i \subset \overline{\mathcal{S}}_g^+$ correspond to loci of curves of compact type.
Goal: Show that $\kappa(\overline{\mathcal{S}}_8^+) = 0$, i.e. this sum of divisors is rigid on $\overline{\mathcal{S}}_8^+$. In so, $\overline{\mathcal{S}}_8^+$ would be the first example of a moduli space of intermediate Kodaira dimension.

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where $\overline{\mathcal{M}}_{8,7}^2 \subset \overline{\mathcal{M}}_8$ is the irreducible locus of plane septics, and $\alpha_i, \beta_i \subset \overline{\mathcal{S}}_g^+$ correspond to loci of curves of compact type.
Goal: Show that $\kappa(\overline{\mathcal{S}}_8^+) = 0$, i.e. this sum of divisors is rigid on $\overline{\mathcal{S}}_8^+$. In so, $\overline{\mathcal{S}}_8^+$ would be the first example of a moduli space of intermediate Kodaira dimension.

- Each component of $K_{\overline{\mathcal{S}}_8^+}$ is a uniruled, rigid, extremal divisor on $\overline{\mathcal{S}}_8^+$.
- Construct a covering curve $\mathfrak{R} \subset \overline{\Theta}_{\mathrm{null}}$ such that

$$\mathfrak{R} \cdot \pi^*(\overline{\mathcal{M}}_{8,7}^2) = 0, \ \mathfrak{R} \cdot \alpha_i = \mathfrak{R} \cdot \beta_i = 0 \ \text{ for } i \geq 1, \ \mathfrak{R} \cdot \overline{\Theta}_{\mathrm{null}} < 0.$$

Then $|nK_{\overline{\mathcal{S}}_8^+}| = 8n \cdot \overline{\Theta}_{\mathrm{null}} + |n(K_{\overline{\mathcal{S}}_8^+} - 8\overline{\Theta}_{\mathrm{null}})|$, and one repeats the procedure and removes $\pi^*(\overline{\mathcal{M}}_{8,7}^2)$ from the canonical system, then the boundary divisors. The most difficult step is the removal of $\overline{\Theta}_{\mathrm{null}}$.

Mukai's model of $\overline{\mathcal{M}}_8$: Fix $V = \mathbb{C}^6$ and

$$\mathbf{G} := G(2, V) \hookrightarrow \mathbf{P}(\wedge^2 V) = \mathbf{P}^{14}$$

the Grassmannian of lines in $\mathbf{P}^5$. Note that $\dim(\mathbf{G}) = 8, K_{\mathbf{G}} = \mathcal{O}_{\mathbf{G}}(-6)$, and that curve linear sections of $\mathbf{G}$ are canonical curves of genus 8.

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Mukai model of $\overline{\mathcal{M}}_8$

$$\mathfrak{M}_8 := G\big(8, \wedge^2 V\big)^{ss} // SL(V)$$

There is a birational map $\phi : \overline{\mathcal{M}}_8 \dashrightarrow \mathfrak{M}_8$, given by $\phi^{-1}(H) := [\mathbf{G} \cap H]$, for a 7-plane $H \subset \mathbf{P}^{14}$. Note that $\rho(\mathfrak{M}_8) = 1$ (whereas $\rho(\overline{\mathcal{M}}_8) = 6$), thus $\mathrm{Exc}(\phi)$ should have 5 irreducible components.

Theorem

The morphism ϕ contracts the boundary divisors $\Delta_1, \ldots, \Delta_4 \subset \overline{\mathcal{M}}_8$. Furthermore, ϕ blows the septic locus $\overline{\mathcal{M}}^2_{8,7}$ down to a point.

Let $[C, \eta] \in \Theta_{\mathrm{null}}$, and $Q_C \in H^0(\mathcal{I}_{C/\mathbf{P}^7}(2))$ rank 3 quadric inducing η. Restriction induces an isomorphism at the level of quadrics:

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There is a $\mathbf{P}^6$ of extensions of the canonical curve C by a $K3$ surface:

$$\begin{array}{ccccc} C & \longrightarrow & S & \longrightarrow & \mathbf{G} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^7 & \longrightarrow & \mathbf{P}^8 & \longrightarrow & \mathbf{P}^{14} \end{array}$$

For each such extension, Q_C lifts to a quadric $Q_S \in H^0(\mathbf{P}^8, \mathcal{I}_{S/\mathbf{P}^8}(2))$.

$$\mathrm{rk}(Q_S) \leq 2 + \mathrm{rk}(Q_C) = 5.$$

Proposition

There exists a $K3$ extension $C \subset S \subset \mathbf{G} \subset \mathbf{P}^{14}$ with $\mathrm{rk}(Q_S) = 4$.

One has a finite covering $f : S \to Q_0 := \mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$. The two projections induce elliptic pencils $|E_1|, |E_2|$ on S (thus $E_1^2 = E_2^2 = 0$), and $C \equiv E_1 + E_2$. Since $g(C) = 8$, it follows $E_1 \cdot E_2 = \deg(f) = 7$.

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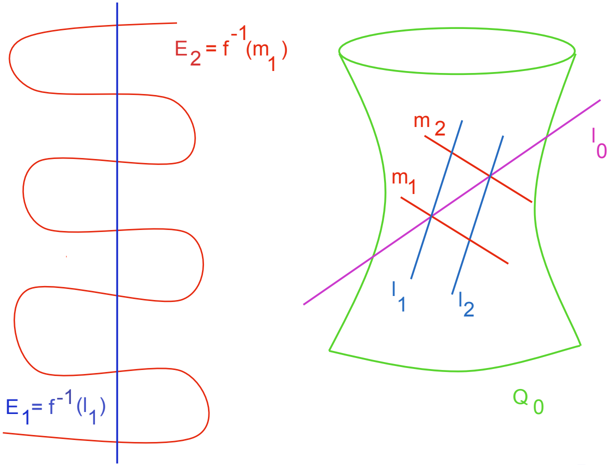
A covering family for $\overline{\Theta}_{\mathrm{null}}$: $\quad f : S \stackrel{7:1}{\to} Q_0$

If $l_0 \subset \mathbf{P}^3$ is a general line, then $\mathfrak{R}$ is induced by planes through l_0:

$$\mathfrak{R} := f^*(\text{planes through } l_0) \subset \overline{\mathcal{S}}_8^+.$$



The numerical characters of the spin family $\mathfrak{R} \subset \overline{\mathcal{S}}_8^+$:

- $\mathfrak{R} \cdot \lambda = \pi_*(\mathfrak{R}) \cdot \lambda = g + 1 = 9$.
- $\mathfrak{R} \cdot (\alpha_0 + 2\beta_0) = \pi_*(\mathfrak{R}) \cdot \delta_0 = 6(g+3) = 66$ (number of singular fibres in a pencil of genus g curves on a $K3$ surface).

The pencil $\mathfrak{R} \subset \overline{\mathcal{S}}_8^+$ contains two singular fibres consisting each of two elliptic curves meeting in 7 points; these correspond to the planes in $\mathbf{P}^3$ containing the rulings through the points $l_0 \cap Q_0$. Each of these counts with multiplicity $7/2$ (the division by 2 because of the branching of β_0). Therefore

$$\mathfrak{R} \cdot \beta_0 = \frac{7}{2} + \frac{7}{2} = 7$$

and then $\mathfrak{R} \cdot \alpha_0 = 52$.

$$\mathfrak{R} \cdot \overline{\Theta}_{\mathrm{null}} = \mathfrak{R} \cdot \left(\frac{\lambda}{4} - \frac{\alpha_0}{16}\right) = \frac{9}{4} - \frac{52}{16} = -1 < 0.$$

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Nikulin surfaces and $\overline{\mathcal{S}}_g^+$

Given a $K3$ surface S and a collection $\{R_j\}_{j=1}^N$ of disjoint rational curves on S, Nikulin asked in 1975, when is there a $2:1$ cover $\widetilde{S} \to S$ branched precisely along $\bigcup_{j=1}^N R_j$? Equivalently, $\exists e \in \mathrm{Pic}(S)$, such that

$$2e = \mathcal{O}_S(R_1 + \cdots + R_N).$$

Answer:

1. $N = 16$ and $\widetilde{S}$ is birational to an abelian variety: Kummer surface.
2. $N = 8$ and $\widetilde{S}$ is again a $K3$ surface: Nikulin surface

We introduce several moduli spaces: the Prym moduli space

$$\mathcal{R}_g := \{[C, \eta] : [C] \in \mathcal{M}_g,\ \eta \in \mathrm{Pic}^0(C),\ \eta^{\otimes 2} = \mathcal{O}_C\}.$$

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$\mathbf{P}^g$-bundle over $\mathcal{N}_g$

$$\mathcal{PN}_g := \{([S, C, e]) : [S, \mathcal{O}_S(C)] \in \mathcal{N}_g, C \subset S\}.$$

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Dimension count:
$\dim(\mathcal{N}_g) = 11 (= 19 - \#\{R_j\}_{j=1}^8)$; $\dim(\mathcal{PN}_g) = 11 + g$.

Question

When is χ_g dominant? A necessary condition is that

$$11 + g \geq 3g - 3 \Leftrightarrow g \leq 7.$$

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$$11 + g \geq 3g - 3 \Leftrightarrow g \leq 7.$$

Theorem

(F, Verra 2009) The general Prym curve $[C, \eta] \in \mathcal{R}_g$ *lies on a Nikulin surface if and only if* $g \le 7, g \ne 6$.

Remark

In genus 6*: the locus* $\mathrm{Im}(\chi_6) \subset \mathcal{R}_6$ *is a divisor, namely the ramification locus of the Prym map* $\mathfrak{Pr} : \mathcal{R}_6 \to \mathcal{A}_5$. *The general Prym curve* $[C, \eta] \in \mathcal{R}_6$*, lies instead on an Enriques surface!*

Theorem

(Mukai) The general $[C] \in \mathcal{M}_g$ *lies on a K3 surface if and only if* $g \le 11$ *and* $g \ne 10$*. In genus* 10*, the locus*

$$\mathcal{K}_{10} := \{[C] \in \mathcal{M}_{10} : C \text{ lies on a } K3 \text{ surface}\}$$

is an irreducible divisor on $\mathcal{M}_{10}$.

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Theorem

(F, Verra 2010) The even spin moduli space $\overline{\mathcal{S}}_g^+$ is uniruled for $g \leq 7$.

Sketch of proof: Start with $[C, \eta] \in \mathcal{S}_g^+$. Choose $e_C \in \mathrm{Pic}^0(C)[2]$ such that $\eta \otimes e_C = \mathcal{O}_C(x_1 + \cdots + x_{g-1})$ is an odd theta-characteristic. Let

$$(C, \eta_C) \subset (S, e) \subset \mathbf{P}^g$$

be a Nikulin extension of C. Consider the pencil of hyperplanes in $\mathbf{P}^g$ through the points $x_1, \ldots, x_{g-1}$:

$$\{H_t\}_{t \in \mathbf{P}^1} := |\mathcal{I}_{\sum_{i=1}^{g-1} x_i / S}(C)|$$

induces a rational covering curve in $\overline{\mathcal{S}}_g^+$:

$$\Big\{[C_t := H_t \cap S,\ e_{C_t} \otimes \mathcal{O}_{C_t}(x_1 + \cdots + x_{g-1})]\Big\}_{t \in \mathbf{P}^1}.$$

Note that each section C_t will be tangent to H_t along the fixed divisor $x_1 + \cdots + x_{g-1}$. So $\overline{\mathcal{S}}_g^+$ is uniruled.

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