# Toward a geometric construction of Fake Projective Planes

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# Outline

Fake Projective Planes
- Definition and known constructions
- Prasad-Yeung's classification

Quotients of fake projective planes

Reverse Construction

$(2, 3)$-elliptic surface case

$(2, 4)$-elliptic surface case

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# Definition

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Such a surface is called *a fake projective plane* if it is not isomorphic to $\mathbb{CP}^2$.

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- $K_X$ of a fpp $X$ is ample. So a fpp is exactly a surface of general type with $p_g(X) = 0$ and $c_1(X)^2 = 3c_2(X) = 9$.
- Its universal cover is the unit 2-ball $\mathbf{B}^2 \subset \mathbb{C}^2$ (Aubin76-Yau77), hence $\pi_1(X)$ is infinite.
- $\pi_1(X)$ is a discrete, torsion-free, cocompact subgroup of $PU(2,1)$. Such ball quotients are strongly rigid (Mostow's rigidity 73), so their moduli space consists of a finite number of points.
- $\pi_1(X)$ has covolume 1 in $PU(2,1)$ (Hirzebruch Proportionality 1958).

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REMARK. For differential topologists, a fake projective plane would mean a simply connected symplectic 4-manifold with the same Betti numbers as $\mathbb{CP}^2$, but not diffeomorphic to $\mathbb{CP}^2$.

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Keum(2006) gave a construction of a fpp by taking a degree 3 cover and then degree 7 cover of a suitable contraction of a $(2, 3)$-elliptic surface, described by Ishida(1988), which is covered by Mumford's fpp.
Both fpp's are degree 21 covers of Ishida's surface, one is Galois, the other is not.

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Both fpp's are degree 21 covers of Ishida's surface, one is Galois, the other is not.

REMARK. Ishida's surface is the ball quotient by a maximal arithmetic subgroup of $PU(2, 1)$ containing torsion elements. It is not known how to construct it geometrically.

# Prasad-Yeung's classification (2007, 2010)

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Description of algebraic groups in which $\Pi$ is arithmetic

- There is a pair $(k, l)$ of number fields, $k$ is totally real, $l$ a totally complex quadratic extension of $k$.
- There is a central simple algebra $D$ of degree 3 with center $l$ and an involution $\iota$ of the second kind on $D$ such that $k = l^{\iota}$.
- The algebraic group $\bar{G}(k) \cong \{z \in D | \iota(z)z = 1\}/\{t \in l | \bar{t}t = 1\}$.
- There is one Archimedean place $\nu_0$ of $k$ so that $\bar{G}(k_{\nu_0}) \cong PU(2,1)$ and $\bar{G}(k_{\nu})$ is compact for all other Arch. places $\nu$.
- The data $(k, l, D, \nu_0)$ determines $\bar{G}$ up to $k$-isomorphism.
- Using Prasad's volume formula, PY eliminated most $(k, l, D, \nu_0)$, making a short list of possibilities where $\Pi$'s might occur, which yields a short list of maximal arithmetic subgroups $\bar{\Gamma}$ which might contain a $\Pi$.

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COROLLARY.

$$Aut(X) \cong N(\pi_1(X))/\pi_1(X),$$

where $N(\pi_1(X))$ is the normalizer of $\pi_1(X)$ in $\bar{\Gamma}$.
In particular,

$$Aut(X) = \{1\},\ \mathbb{Z}/3\mathbb{Z},\ (\mathbb{Z}/3\mathbb{Z})^2,\ 7:3.$$

# Cartwright and Steger's computation (2010)

- There are exactly 28 $\bar{\Gamma}$'s (or 28 classes).
- There are exactly 50 $\Pi$'s. Each corresponds to two fpp's, complex conjugate to each other.
- There are exactly 100 fpp's.
- 39 of the 50 have $\mathrm{Aut}(X) \neq \{1\}$.

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We classified all possible structures of the quotient surface $X/G$ and its minimal resolution (Keum 2008).

1. If $G = \mathbb{Z}/3\mathbb{Z}$, then $X/G$ is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 3$.
2. If $G = (\mathbb{Z}/3\mathbb{Z})^2$, then $X/G$ is a $\mathbb{Q}$-homology projective plane with 4 singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 1$.
3. If $G = \mathbb{Z}/7\mathbb{Z}$, then $X/G$ is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{7}(1,5)$ and its minimal resolution is a $(2,3)$-, $(2,4)$-, or $(3,3)$-elliptic surface.
4. If $G = 7:3$, then $X/G$ is a $\mathbb{Q}$-homology projective plane with 4 singular points, 3 of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{7}(1,5)$, and its minimal resolution is a $(2,3)$-, $(2,4)$-, or $(3,3)$-elliptic surface.

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Here, a $\mathbb{Q}$-homology projective plane is a normal projective surface with the same Betti numbers as $\mathbb{P}^2$.
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An $(a, b)$-elliptic surface is a relatively minimal elliptic surface over $\mathbb{P}^1$ with two multiple fibres of multiplicity $a$ and $b$ respectively.
It has Kodaira dimension 1 if and only if
$a \geq 2, b \geq 2, a + b \geq 5$.
It is an Enriques surface iff $a = b = 2$, and it is rational iff $a = 1$ or $b = 1$.
All $(a, b)$-elliptic surfaces have $p_g = q = 0$.

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# Reverse Construction

Given a $\mathbb{Q}$-homology projective plane satisfying one of the descriptions 1-4, can one construct a fpp by taking a suitable cover, or a composition of two suitable covers?
In other words, do the descriptions (1)-(4) above characterize the quotients of fake projective planes?

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## Theorem

*Let $Z$ be a $\mathbb{Q}$-homology projective plane satisfying one of the descriptions (1)-(4) above. Assume that $H_1(Z, \mathbb{Z})$ has no element of order 3. Then a fpp can be constructed from $Z$.*

# Outline of Proof

By a lattice theory, the basket of singularities implies the existence of a suitable cover, or a composition of two suitable covers branched at the singularities, yielding a nonsingular surface $X$.

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The case $p_g = q = 1$ can be eliminated by considering the Albenese fibration, and by Holomorphic Lefschetz and Topological Lefschetz applied to an automorphism $\sigma$ of $X$ of order 3 or 7 with such fixed points.

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The case $p_g = q = 2$ was eliminated by Yeung.

# Fundamental group of a quotient $X/G$





Write the group $G \cong \tilde{G}/\pi_1(X)$, where $\pi_1(X) < \tilde{G} < \bar{\Gamma}$. Then

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These groups have been computed by Cartwright and Steger. According to their computation (unpublished),

$\pi_1(X/G) = \{1\}$ or $\mathbb{Z}/2\mathbb{Z}$, if $G = \mathbb{Z}/7\mathbb{Z},\ (\mathbb{Z}/3\mathbb{Z})^2$ or $7:3$.

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In particular, $(3,3)$-elliptic surface does not occur in my list.

# Outline

# $(2,3)$-elliptic surface case

## Theorem

*Let $Z$ be a $\mathbb{Q}$-homology projective plane with $4$ singular points, 3 of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{7}(1,5)$. Assume that its minimal resolution $V$ is a $(2,3)$-elliptic surface.*

1. *There is a triple cover $Y' \to Z$ branched at the three singular points of type $\frac{1}{3}(1,2)$, and $Y'$ is a $\mathbb{Q}$-homology projective plane with $3$ singular points of type $\frac{1}{7}(1,5)$. The minimal resolution $Y$ of $Y'$ is a $(2,3)$-elliptic surface, and every fibre of $V$ does not split in $Y$.*
2. *The elliptic fibration on $V$ has $4$ singular fibres of type $I_3$, some of them may be a multiple fibre.*
3. *The elliptic fibration on $Y$ has $4$ singular fibres of type $\mu I_9 + \mu_1 I_1 + \mu_2 I_1 + \mu_3 I_1$.*

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1. *There is a triple cover $Y' \to Z$ branched at the three singular points of type $\frac{1}{3}(1,2)$, and $Y'$ is a $\mathbb{Q}$-homology projective plane with $3$ singular points of type $\frac{1}{7}(1,5)$. The minimal resolution $Y$ of $Y'$ is a $(2,3)$-elliptic surface, and every fibre of $V$ does not split in $Y$.*
2. *The elliptic fibration on $V$ has $4$ fibres of type $I_3$, some of them may be a multiple fibre, and the fibre containing two $(-2)$-curves lying over the singularity of type $\frac{1}{7}(1,5)$ has multiplicity $\leq 2$.*
3. *The elliptic fibration on $Y$ has $4$ singular fibres of type $\mu I_9 + \mu_1 I_1 + \mu_2 I_1 + \mu_3 I_1$ with $\mu \leq 2$.*