The automorphism group of $\overline{M}_{0,n}$

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joint with Andrea Bruno

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Today I am interested in the Automorphisms of $\overline{M}_{0,n}$.

Fulton's Conjecture

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Conjecture (Fulton)

when $n \geq 5$ these are the only automorphisms

Kapranov's construction

$$\overline{M}_{0,n} \cong \overline{\left\{ \begin{array}{c} \text{rational normal curves in } \mathbb{P}^{n-2} \\ \text{through } n \text{ general points} \\ \{q_1, \dots, q_n\} \end{array} \right\}} =: H_q$$

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fixing one of the points, say q_1, the general curve is uniquely determined by its tangent at q_1.

This gives a birational map

$$\chi : \mathbb{P}^{n-3} \dashrightarrow \overline{M}_{0,n}$$

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where the domain represents the directions through q_1, considering reducible curves in H_q it is easy to see that χ is not defined along the linear spaces spanned by the $(n-1)$ points associated to the lines $\langle q_1, q_j\rangle$ and it is defined elsewhere.

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obtained via blowing up on a *"dimension increasing"* order the linear spaces spanned by $n-1$ points in general position in $\mathbb{P}^{n-3}$.

Definition

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$$f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$$

with $I \cup \{i\} = \{1, \ldots, n\}$, obtained via the iterated blow up described before, based on points of $\mathcal{K}$.

Standard Cremona Transformations

$$\mathbb{P}^{n-3} \xleftarrow{\ f_i\ } \overline{M}_{0,n} \xrightarrow{\ f_j\ } \mathbb{P}^{n-3}$$

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ω_{ij} is the standard Cremona transformation, centered on $\mathcal{K} \setminus \{p_j\}$, i.e. $(x_0, \dots, x_{n-3}) \mapsto (x_0^{-1}, \dots, x_{n-3}^{-1})$.

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Let us work out special cases particularly meaningful for us.

Forgetful maps

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$$\begin{array}{ccc}
\overline{M}_{0,n} & \xrightarrow{\phi_I} & \overline{M}_{0,n-|I|} \\
\big\downarrow{\scriptstyle f_j} & & \big\downarrow{\scriptstyle f_h} \\
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Remark

The fibers of a map forgetting one marking are either lines through a point in $\mathcal{K}$ or RNC through $\mathcal{K}$.

Permutations

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$\Sigma = \Omega \circ \omega_{12}$ where Ω is a projectivity.

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Remark

Lines through the points in the Kapranov set $\mathcal{K}$ are sent to either lines or RNC through $\mathcal{K}$

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We approach this question studying more generally the diagrams

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{f} & \overline{M}_{0,r} \\ \downarrow f_j & & \downarrow f_h \\ \mathbb{P}^{n-3} & \overset{\phi}{\dashrightarrow} & \mathbb{P}^{r-3} \end{array}$$

Pencils on $\overline{M}_{0,n}$

The first step is to consider a morphism with connected fibers

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let $\mathcal{L}=f^*(\mathcal{O}(1))$ and $\mathcal{L}_i=f_{i*}\mathcal{L}\subset|\mathcal{O}(d_i)|$. Then $\mathcal{L}_i$ is a pencil of hypersurfaces without fixed components and with a very special Base Locus.

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- there is a choice of $(n-3)$ points in $\mathcal{K}$, say $\{p_{j_1}, \ldots, p_{j_{n-3}}\}$, such that $\mathcal{L}_{i|\langle p_{j_1},\ldots,p_{j_{n-3}}\rangle}$ is a pencil without fixed components;

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With these properties it is easy to prove the following Theorem by induction on n.

Theorem

Let $f : \overline{M}_{0,n} \to \mathbb{P}^1$ *be a morphism then* f *can be factored via a forgetful map* $\phi_I : \overline{M}_{0,n} \to \mathbb{P}^1$.

Theorem

Let $f : \overline{M}_{0,n} \to \mathbb{P}^1$ be a morphism then f can be factored via a forgetful map $\phi_I : \overline{M}_{0,n} \to \mathbb{P}^1$.

This, plus a bit more work allows to prove, by induction on r, the following.

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This, plus a bit more work allows to prove, by induction on r, the following.

Theorem

Let $f : \overline{M}_{0,n} \to \overline{M}_{0,r}$ be a morphism with connected fibers then f, up to an automorphism of $\overline{M}_{0,r}$, is a forgetful map $\phi_J : \overline{M}_{0,n} \to \overline{M}_{0,r}$.

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Let $g \in \mathrm{Aut}(\overline{M}_{0,n})$ be an automorphism, and $\phi_i : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ the map forgetting the i-th marking. By our result $\phi_i \circ g$ is associated to a map forgetting a marking, say j_i.

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This produces a morphism

$$\chi : \mathrm{Aut}(\overline{M}_{0,n}) \to S_n$$

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This produces a morphism

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given by

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$$\phi_i \circ g$$

is associated to the forgetful map forgetting the i-th marking.

Let us look at it from the viewpoint of $\mathbb{P}^{n-3}$.

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We have a birational self map $\gamma_n : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3}$, induced by $\mathcal{H}_n \subset |\mathcal{O}(d)|$ and a Kapranov set $\mathcal{K} = \{p_1, \ldots, p_{n-1}\}$ such that

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$$(n-3)d - \sum_{i=1}^{n-1}\mathrm{mult}_{p_i}\mathcal{H}_n = n-3.$$

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hence $d=1$ and γ_n is a projectivity fixing $n-1$ general points.

This is enough to prove that γ_n and henceforth g are the identity, giving the required

Theorem (Fulton's Conjecture)

$Aut(\overline{M}_{0,n}) \cong S_n$, for $n \geq 5$.

With these ideas we are able to study other special classes of fiber type morphisms from $\overline{M}_{0,n}$, for either low n or low dimensional image or linear fibers...