



*joint with Andrea Bruno*

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Today I am interested in the Automorphisms of  $\overline{M}_{0,n}$ .

# Fulton's Conjecture

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Conjecture (Fulton)

*when  $n \geq 5$  these are the only automorphisms*

# Kapranov's construction

$$\overline{M}_{0,n} \cong \overline{\left\{ \begin{array}{l} \text{rational normal curves in } \mathbb{P}^{n-2} \\ \text{through } n \text{ general points} \\ \{q_1, \dots, q_n\} \end{array} \right\}} =: H_q$$

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fixing one of the points, say  $q_1$ , the general curve is uniquely determined by its tangent at  $q_1$ .



This gives a birational map

$$\chi : \mathbb{P}^{n-3} \dashrightarrow \overline{M}_{0,n}$$

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where the domain represents the directions through  $q_1$ , considering reducible curves in  $H_q$  it is easy to see that  $\chi$  is not defined along the linear spaces spanned by the  $(n-1)$  points associated to the lines  $\langle q_1, q_j \rangle$  and it is defined elsewhere.

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obtained via blowing up on a “dimension increasing” order the linear spaces spanned by  $n - 1$  points in general position in  $\mathbb{P}^{n-3}$ .

## Definition

A Kapranov set  $\mathcal{K} \subset \mathbb{P}^{n-3}$  is a set of  $(n-1)$  linearly independent points in  $\mathbb{P}^{n-3}$ , labelled by a subset  $I \subset \{1, \dots, n\}$ .

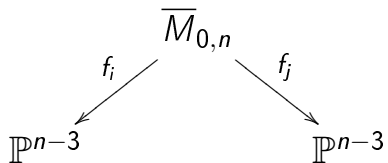
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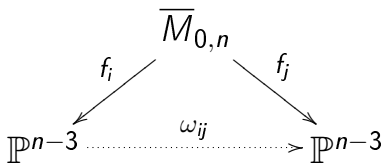
with  $I \cup \{i\} = \{1, \dots, n\}$ , obtained via the iterated blow up described before, based on points of  $\mathcal{K}$ .

# Standard Cremona Transformations

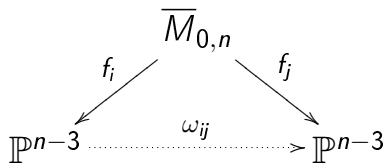




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$\omega_{ij}$  is the standard Cremona transformation, centered on  $\mathcal{K} \setminus \{p_j\}$ , i.e.  $(x_0, \dots, x_{n-3}) \mapsto (x_0^{-1}, \dots, x_{n-3}^{-1})$ .

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Let us work out special cases particularly meaningful for us.

# Forgetful maps



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$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{\phi_I} & \overline{M}_{0,n-|I|} \\ \downarrow f_j & & \downarrow f_h \\ \mathbb{P}^{n-3} & & \mathbb{P}^{n-|I|-3} \end{array}$$

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## Remark

The fibers of a map forgetting one marking are either lines through a point in  $\mathcal{K}$  or RNC through  $\mathcal{K}$ .

# Permutations

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Lines through the points in the Kapranov set  $\mathcal{K}$  are sent to either lines or RNC through  $\mathcal{K}$

Our plan for Fulton's Conjecture is to prove that this Remark is true for an arbitrary automorphism  $g \in \text{Aut}(\overline{M}_{0,n})$ .

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We approach this question studying more generally the diagrams

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{f} & \overline{M}_{0,r} \\ \downarrow f_j & & \downarrow f_h \\ \mathbb{P}^{n-3} & \xrightarrow{\phi} & \mathbb{P}^{r-3} \end{array}$$

# Pencils on $\overline{M}_{0,n}$

The first step is to consider a morphism with connected fibers

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let  $\mathcal{L} = f^*(\mathcal{O}(1))$  and  $\mathcal{L}_i = f_{i*}\mathcal{L} \subset |\mathcal{O}(d_i)|$ . Then  $\mathcal{L}_i$  is a pencil of hypersurfaces without fixed components and with a very special Base Locus.

Using Kapranov's maps and the description of Cremona Transformations it is possible to prove the following properties of  $\mathcal{L}_i := \{A_1, A_2\}$ :

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- there is a choice of  $(n - 3)$  points in  $\mathcal{K}$ , say  $\{p_{j_1}, \dots, p_{j_{n-3}}\}$ , such that  $\mathcal{L}_i|_{\langle p_{j_1}, \dots, p_{j_{n-3}} \rangle}$  is a pencil without fixed components;

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With these properties it is easy to prove the following Theorem by induction on  $n$ .

## Theorem

*Let  $f : \overline{M}_{0,n} \rightarrow \mathbb{P}^1$  be a morphism then  $f$  can be factored via a forgetful map  $\phi_I : \overline{M}_{0,n} \rightarrow \mathbb{P}^1$ .*



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This, plus a bit more work allows to prove, by induction on  $r$ , the following.

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This, plus a bit more work allows to prove, by induction on  $r$ , the following.

## Theorem

*Let  $f : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r}$  be a morphism with connected fibers then  $f$ , up to an automorphism of  $\overline{M}_{0,r}$ , is a forgetful map  $\phi_J : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r}$ .*

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Let  $g \in \text{Aut}(\overline{M}_{0,n})$  be an automorphism, and  $\phi_i : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$  the map forgetting the  $i$ -th marking. By our result  $\phi_i \circ g$  is associated to a map forgetting a marking, say  $j_i$ .

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given by

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$$\phi_i \circ g$$

is associated to the forgetful map forgetting the  $i$ -th marking.

Let us look at it from the viewpoint of  $\mathbb{P}^{n-3}$ .

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We have a birational self map  $\gamma_n : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3}$ ,  
induced by  $\mathcal{H}_n \subset |\mathcal{O}(d)|$  and a Kapranov set  
 $\mathcal{K} = \{p_1, \dots, p_{n-1}\}$  such that

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$$(n - 3)d - \sum_{i=1}^{n-1} \text{mult}_{p_i} \mathcal{H}_n = n - 3.$$



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This is enough to prove that  $\gamma_n$  and henceforth  $g$  are the identity, giving the required

**Theorem (Fulton's Conjecture)**

$$\text{Aut}(\overline{M}_{0,n}) \cong S_n, \text{ for } n \geq 5.$$

With these ideas we are able to study other special classes of fiber type morphisms from  $\overline{M}_{0,n}$ , for either low  $n$  or low dimensional image or linear fibers...