Enriques surfaces of type E_7

$S = X/\varepsilon$ Enriques surface

$\mathbb{Z}_S^\omega$ local system $\longleftrightarrow$ non-trivial rep.
$\pi_1 \xrightarrow{\sim} \{\pm 1\} \subset \mathrm{Aut}\,\mathbb{Z}$

$\mathbb{Z}_S^\omega \otimes \mathcal{O}_S^{an} = \mathcal{O}_S^{an}(K_S)$

$$\begin{cases} H := H^2(S, \mathbb{Z}_S^\omega) \cong \mathbb{Z}^{12} \\ H \times H \longrightarrow \mathbb{Z} \quad \text{induced by } \mathbb{Z}_S^\omega \times \mathbb{Z}_S^\omega \to \mathbb{Z}_S \\ H \otimes_{\mathbb{Z}} \mathbb{C} \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \quad \dim = (1, 10, 1) \end{cases}$$

polarized Hodge structure of wt 2

Torelli type theorem (in new formulation)

S, S' two Enriques surfaces

$H^2(S, \mathbb{Z}_S^\omega) \cong H^2(S', \mathbb{Z}_{S'}^\omega)$ as pol. Hodge str.

$\Longrightarrow S \cong S'$

$\mathrm{Pic}^\omega S := H^{1,1} \cap H^2(S, \mathbb{Z}_S^\omega)$ twisted Picard lattice

$$H^2(S, \mathbb{Z}_S^{w}) = \mathrm{Ker}\,[H^2(X,\mathbb{Z}) \longrightarrow H^2(S,\mathbb{Z})]$$

$$H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{22}, \qquad H^2(S,\mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2$$

$H \cong (\mathrm{Ker})(\frac{1}{2})$ as lattice

$H \cong \langle 1 \rangle^2 + \langle -1 \rangle^{10}$ odd unimodular, $\mathrm{sgn} = (2, 10)$

$\mathrm{Pic}^{w} S = \mathrm{Ker}\,[\mathrm{Pic}\,X \longrightarrow \mathrm{Pic}\,S](\frac{1}{2})$

neg def lattice of rk $\rho(X) - 10$

<u>Def.</u> L : neg. def. (integral) lattice

S is of (lattice) type L

$\Leftrightarrow$ $\exists$ primitive embedding

$$L \hookrightarrow \mathrm{Pic}^{w} S$$

Assume that

(*) prim. emb. $L \hookrightarrow \langle 1 \rangle^{~~3~~\,2} + \langle -1 \rangle^{~~9~~\,10}$

is unique,

and let $L^{\perp}$ be the orthogonal complement.

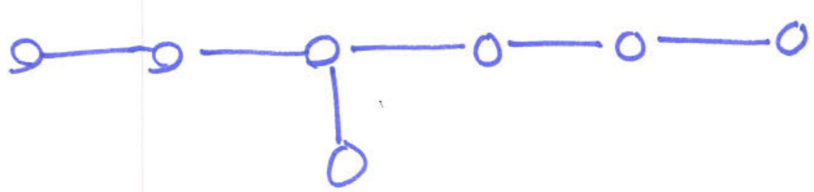
Then the period map

$$\left\{ \begin{array}{l} \text{Enriques surface} \\ \text{of type } L \end{array} \right\} \Big/ \text{isom} \xrightarrow{\pi} D^{10-r} \Big/ O(L^{\perp})$$

is injective by Torelli type theorem.

$$\begin{cases} r = \operatorname{rk} L \\ D^{10-r} = \left\{ z \in L^{\perp} \otimes_{\mathbb{Z}} \mathbb{C} \;\middle|\; \begin{array}{l} (z, z) = 0 \\ (z, \bar{z}) > 0 \end{array} \right\} \\ \quad \text{bdd sym. domain of type IV} \\ \quad \text{of dim. } 10 - r \\ O(L^{\perp}) \quad \text{orthogonal group of } L^{\perp} \end{cases}$$

We study the case $L = E_7$, the (negative) root lattice of Dynkin diagram



$|\operatorname{disc}| = 2$

(Complement of (−2)-vector in E_8.)

$r=7$, E_7 satisfies $(*)$ and

$$L^\perp \cong \langle 1\rangle^2 + \langle -1\rangle^2 + \langle -2\rangle.$$

domain of type IV

$$D^3 \cong \mathfrak{H}_2 = \left\{ \tau = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \,\middle|\, \operatorname{Im}\tau > 0 \right\}$$

Siegel upper half space of degree 2

Lemma (i) $D^3 / O(E_7^\perp) \cong \mathfrak{H}_2 / \Gamma_0^*(2)$

$$\Gamma_0(2) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z}) \,\middle|\, \begin{matrix} C \equiv 0 \\ \text{mod } 2 \end{matrix} \right\}$$

$$\Gamma_0^*(2) = \Gamma_0(2) \amalg \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & I_2 \\ 2I_2 & 0 \end{pmatrix} \Gamma_0(2)$$

$$\subset Sp(4, \mathbb{R})$$

(ii) $\mathfrak{H}_2 / \Gamma_0(2) \cong \left\{ (A, G) \,\middle|\, G \subset A_{(2)} \right\}$ — A: principally polarized abelian surface, G: Göpel subgp / isom.

$\downarrow$ 2:1

$\mathfrak{H}_2 / \Gamma_0^*(2)$ quotient by Hecke involution

Def. (Göpel)

$|G| = 4$ & Restriction of $A_{(2)} \times A_{(2)} \to \mu_2$ to G is trivial

$(A, G) \rightsquigarrow A/G$ is also p.p. &

$A_{(2)}/G$ is also a Göpel.

$$(A, G) \xleftrightarrow[\text{corresp.}\ (\text{involution})]{\text{Hecke}} \left(A/G,\ A_{(2)}/G\right)$$

<u>Problem</u> Construct an Enriques surface S of type E_7 from (A, G) corr. to $\Pi(S)$. (its period)

$$\begin{array}{ccccc}
 & & (A,G) \in \mathfrak{h}_2/\Gamma_0(2) & & \\
 & & \downarrow & & \Big\downarrow 2:1 \\
S & \longmapsto & \Pi(S) & & \\
\cap & & & \in & \\
\left\{\begin{array}{l}\text{Enriques surface}\\ \text{of type } E_7\end{array}\right\}\Big/\text{isom.} & & \longrightarrow & & D^3/O(E_7^{\perp})
\end{array}$$

We answer using Richelot isogeny when A or A/G is not of product type.

(Product case is limit case and easier.)

Theorem (Richelot, 1830's)

$$A = \mathrm{Jac}\, C, \quad A/G = \mathrm{Jac}\, C'$$

C, C' : curves of genus 2

$\{p_1, \ldots, p_6\} \subset C$, $\{p_1', \ldots, p_6'\} \subset C'$ Weierstrass points

$$G = \{0, p_1 + p_4 - K_C, p_2 + p_5 - K_C, p_3 + p_6 - K_C\}$$

$$G' = \{0, p_1' + p_4' - K_{C'}, p_2' + p_5' - K_{C'}, p_3' + p_6' - K_{C'}\}$$

$$\Phi = \Phi_{|2K_C|} : C \xrightarrow{2:1} Q \subset \mathbb{P}^2$$

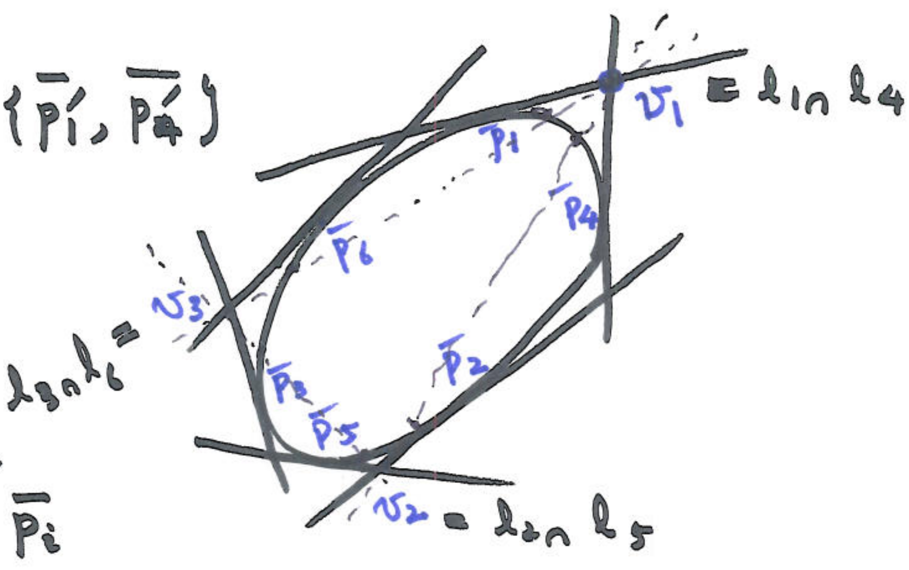
(Q: conic)

$\bar{p}_i := \Phi(p_i)$, $i = 1, \ldots, 6$.

$\Rightarrow$ C' is double cover of Q with branch $\triangle v_1 v_2 v_3 \cap Q$.

$\overline{v_2 v_3} \cap Q = \{\bar{p}_1, \bar{p}_4\}$

etc.



l_i : tangent line of Q at $\bar{p}_i$

Main Theorem

$(A, G) \overset{\text{Hecke}}{\underset{\text{Richelot}}{\longleftrightarrow}} (A', G')$

$\searrow \swarrow$

$\downarrow$

Enriques of type E_7 $\quad S \longmapsto \Pi(S')$

$C, \overline{p}_1, \ldots, \overline{p}_6 \in Q \subset \mathbb{P}^2$ as in

Richelot's thm.

$$\left\{\begin{array}{l} q_1 = l_1 + l_4 \\ q_2 = l_2 + l_5 \\ q_3 = l_3 + l_6 \end{array}\right.$$

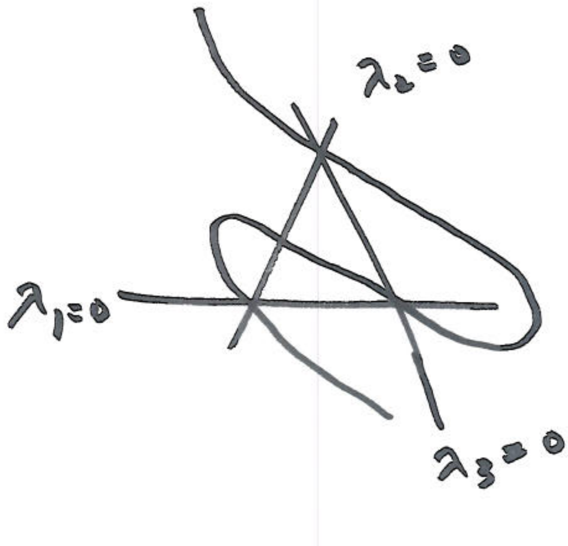
red. conics singular at v_i

$\Rightarrow \tilde{S}$ = double $\mathbb{P}^2_\lambda$ with branch

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \underbrace{\det(\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3)}_{\text{Invariant (cubic)}} = 0$$

of net $\langle q_1, q_2, q_3 \rangle$

of conics



($\tilde{S}$ is a K3 of deg. 2)

w. 3 D_4-sing's.

$S = \tilde{S} \Big/ \left(\begin{array}{l} \text{Involution induced by} \\ \text{Cremona } (\lambda_1 : \lambda_2 : \lambda_3) \\ \quad \mapsto (1/A_1\lambda_1 : 1/A_2\lambda_2 : 1/A_3\lambda_3) \end{array} \right)$

for $\exists$ $A_1, A_2, A_3 \in \mathbb{C}$

Proof $\begin{cases} \text{Torelli type theorem} \\ \text{Auxiliary family of Enriques surfaces} \\ \text{8 - 2 isogeny} \end{cases}$

① $Y_8 = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5$ K3 surface of degree 8

c.i. of 3 quadrics

$X_2 \longrightarrow \mathbb{P}^2_\lambda$ $\quad \tau^2 = \det(\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3)$ ← hom. pol. of deg 2.

K3 surface of degree 2

$H^2(X_2, \mathbb{Z}) \cong v^\perp / \mathbb{Z} v \qquad v = (2, h, 2)$

in $\mathbb{Z} \oplus H^2(Y_8, \mathbb{Z}) \oplus \mathbb{Z}$

② Enriques surfaces of type $D_6 + A_1$

($\tilde{E}_7$ deleted one vertex)

$L := D_6 + A_1 \subset E_7$ index 2

Period map

$$\left\{ \begin{matrix} \text{Enriques} \\ \text{of type} \\ D_6 + A_1 \end{matrix} \right\} \Big/ \text{isom.} \xrightarrow{\ \pi\ } \mathcal{D}^3 / O(L^\perp) \cong \mathfrak{h}_2 / \Gamma_0(2)$$

$$\cong \{(A, G)\} / \text{isom.}$$

π is injection since

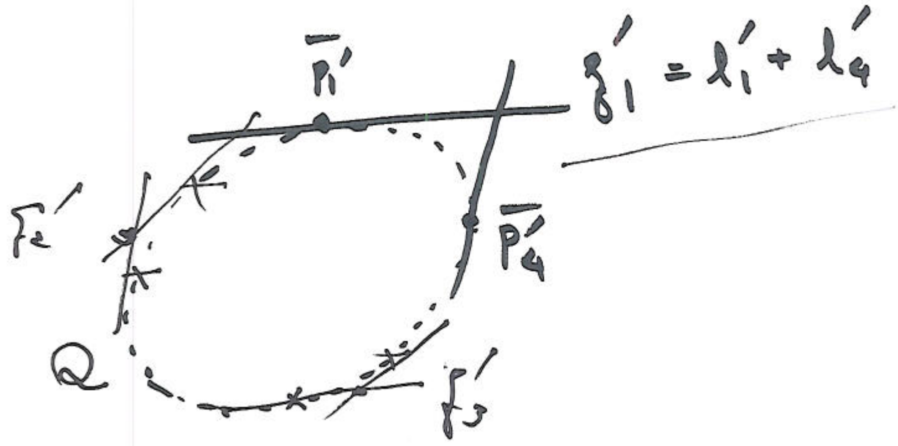
L also satisfies (*)

$$L^\perp \cong \langle 1 \rangle^2 + \langle -2 \rangle^3.$$

$$S \qquad \pi(S) \longleftrightarrow (A, G) \overset{\text{Hecke}}{\underset{\text{Richelot}}{\longleftrightarrow}} (A', G')$$

<u>Theorem</u> Enr. S (of type $D_6 + A_1$) corr. to (A, G) is $\mathbb{Z}/2 \times \mathbb{Z}/2$-covering with branch 3 conics ass. with (A', G').

$D_6 + A_1$



$$\tilde{S} : \; u_1^2 = q_1' , \quad u_2^2 = q_2' , \quad u_3^2 = q_3'$$

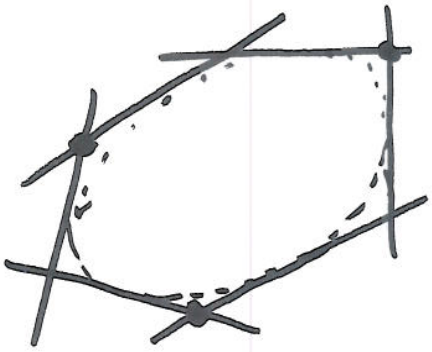
$$S = \tilde{S} \Big/ (u_1, u_2, u_3) \sim (-u_1, -u_2, -u_3)$$

$Y_8 = \tilde{S}$ is c.i. of 3 quadrics in $\mathbb{P}^5$

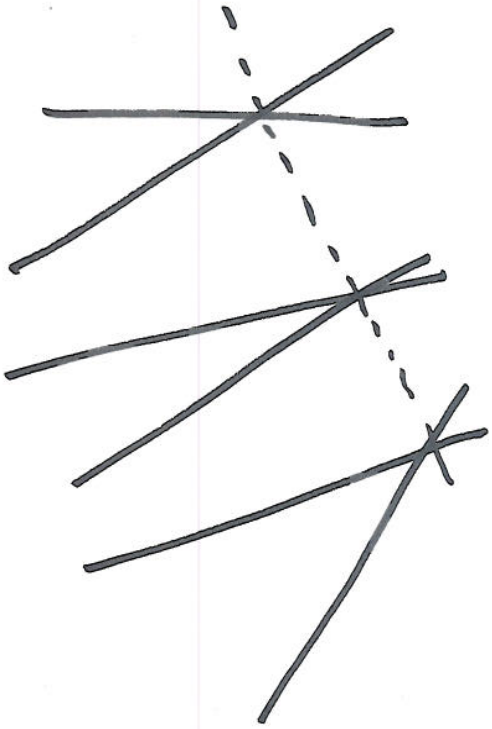
X_2, isogenous to Y_8, (8-2 isogeny) is the K3 cover of the Enriques surface of type E_7 by Torelli and 8-2 isogeny.

Remark

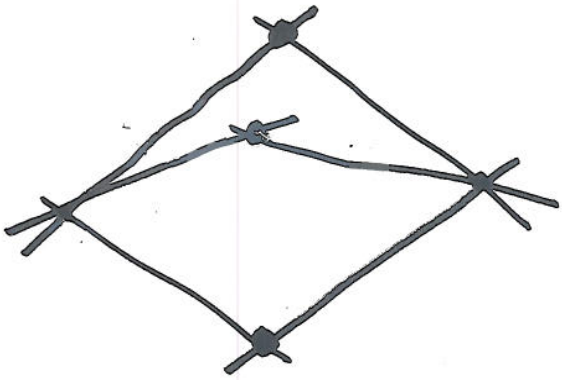
$\mathbb{Z}/2 \times \mathbb{Z}/2$ - cover of $\mathbb{P}^2$



$D_6 + A_1$



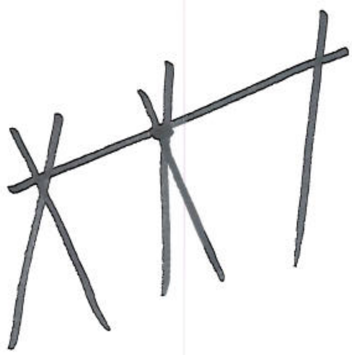
another description of an Enr. of type E_7



type D_8

Enriques of Lieberman type

two triple points



type E_8

studied by Horikawa, Barth - Peters, Mu-Namikawa in 80's

Remark (1) Nikulin–Kondo '80s

S Enriques, $|\mathrm{Aut}(S)| < +\infty$

$\Rightarrow \mathrm{Pic}^{\omega} S \overset{\text{prim.}}{\supset} E_8 + A_1,$ } rk 9

$\Leftarrow$? $D_9,$

$D_8 + A_1 + A_1,$ } rk 8

$D_5 + D_5,$

$E_7 + A_2 + A_1,$

$E_6 + A_4$ or

$A_9 + A_1$

In the last two cases $\mathrm{Aut}\, S = \mathfrak{S}_5$.

(2) $\exists$ Enriques surface S of type

$A_5 + A_5$ s.t.

$\mathrm{Aut}_N S \supset \mathfrak{A}_6,\ 3^2 D_8$

alternative   normalizer of 3-Sylow in $\mathfrak{S}_6$

("N" means symplectic.)