Varieties n–covered by curved of degree δ

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- $\dim(X) = r + 1 \iff X = X^{r+1}$.

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- if we put restrictions on C natural obstructions appear.
- $n = 2$ & C rational curve $\Longleftrightarrow$ X **rationally connected variety** (dubbed *RC*).

Definitions and notation

Theorem (Kollár–Miyaoka–Mori)

1. X *RC* $\iff$ $\exists C = C_{p_1,\ldots,p_n} \subseteq X$ *rational curve passing through* $n \geq 2$ *general points* $p_1, \ldots, p_n \in X$

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3. X^{r+1} RC SMOOTH $r \geq 2$ C any SMOOTH curve $\implies$ $\exists f : C \to X$ EMBEDDING : $p_1, \dots, p_n \in f(C)$

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In this case we shall use the notation :

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- $X = X^{r+1}(n, n-1) \subset \mathbb{P}^N \iff$ $N = r+n-1$, $X^{r+1} \subset \mathbb{P}^{r+n-1}$, $\deg(X) = n-1$ (**minimal degree**)

Pirio–Trepréau bound

Theorem (Pirio–Trepréau, 2009)

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is the **Castelnuovo–Harris function** *bounding the genus* $g(V)$ *of an irreducible variety*

$$V^r \subset \mathbb{P}^{r+n-1}$$

of degree $\deg(V) = d$.

Previously known results and some classifications

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  Scorza (1909) : classification $X = X(2,2) \subset \mathbb{P}^N$ in some cases reconsidered in [Chiantini, Ciliberto, —;201 ?].

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  (b) $N = \binom{r+1+\delta}{r+1} - 1 \Longleftrightarrow X = \nu_\delta(\mathbb{P}^{r+1}) \subset \mathbb{P}^{\binom{r+1+\delta}{2}-1}$

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- $X = X^{r+1}(3,3) \subset \mathbb{P}^{2(r+1)+1} \Longrightarrow$ CLASSIFIED : (Pirio, —; 2010) object of this talk

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- $T = T_x X = \mathbb{P}^{r+1}$ projective tangent space to X at x ;
- $$\pi_T : X \dashrightarrow X_T \subseteq \mathbb{P}^{N-r-2}$$
  projection of X from T, not defined along $T \cap X$.

- $p_1 \neq p_2 \in X$ general $\Longrightarrow \pi_T(p_1) \neq \pi_T(p_2)$.

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  that is

$$X_T = X(r+1, 2, 1) = \mathbb{P}^{r+1-\sigma} = \mathbb{P}^{N-r-2}.$$

In conclusion $N = 2r + 3 - \sigma$, $\sigma \geq 0$ and $\overline{\pi}(r, 3, 3) = 2r + 4$. $\square$

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- $\pi_T^{-1} = \phi_\Lambda : \mathbb{P}^{r+1} \dashrightarrow X \subset \mathbb{P}^{2r+3}$, $\Lambda \subset |\mathcal{O}_{\mathbb{P}^{r+1}}(3)|$ complete linear system.

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  $(\pi_T^{-1}(\Pi_x) = x$ & $\pi_T^{-1}(L)$ is smooth at x for L general).

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$X = X^{r+1}(3,3) \subset \mathbb{P}^{2(r+1)+1}$. Then :

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2. $B_x = Hilb^{t+1}(X,x) \sim_{proj} \tilde{B}_x$ *(usually only* $Hilb^{t+1}(X,x) \subseteq B_x$ *holds !)*
3. X *SMOOTH* $\Longrightarrow$ B_x *et* $\tilde{B}_x$ *are SMOOTH*

From $\mathbf{Bir}_{(2,2)}(\mathbb{P}^r)$ to $X^{r+1}(3,3) \subset \mathbb{P}^{2(r+1)+1}$

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through a general point $q \in \mathbb{P}^{2r+3}$ there passes a unique secant line to X.

Remark

The Structure Theorem has important applications to the classification of (SMOOTH) OADP varieties.

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Cubic Jordan algebras

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$$X_{\mathbb{C}} = \overline{\{(1 : x : x^2 : x^3), \quad x \in \mathbb{C}\}} \subset \mathbb{P}^3.$$

2. by the previous construction

$$X_J = X^{r+1}(3,3) \subset \mathbb{P}^{2(r+1)+1}.$$

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$\exists$ *a Jordan algebra structure* $J = (\mathbb{C}^{r+1}, *)$ *such that*

$$\phi = \bullet^{\#}.$$

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Every $X = X^{r+1}(3,3) \subset \mathbb{P}^{2(r+1)+1}$ is projectively equivalent to a X_J for some Jordan algebra J.

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$$J_1 = \mathbb{C} \times \mathbb{C} \times \mathbb{C} \,, \qquad J_2 = \mathbb{C} \times \frac{\mathbb{C}[X]}{(X^2)} \,, \qquad J_3 = \frac{\mathbb{C}[X]}{(X^3)} \,.$$

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Algbre	**Adjoint** $(x_1, x_2, x_3, x_4)^{\#}$
$\mathbb{C} \times \mathbb{C}[X,Y] \,/\, (X,Y)^2$	$(x_2{}^2\,,\, x_1x_2\,,\, -x_1x_3\,,\, -x_1x_4)$
$\mathbb{C}[X,Y]/\,(X^2, Y^2)$	$(x_1{}^2\,,\, -x_1x_2\,,\, -x_1x_3\,,\, 2x_2x_3 - x_1x_4)$
$\mathbb{C}[X,Y]/\,(X^3, XY, Y^2)$	$(x_1{}^2\,,\, -x_1x_2\,,\, -x_1x_3\,,\, x_2{}^2 - x_1x_4)$
$\mathbb{C} \times \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$	$(x_2\, x_4\,,\, x_1x_4\,,\, -x_1x_3\,,\, x_1x_2)$
$\left\{ \begin{pmatrix} a & 0 & 0 \\ c & a & 0 \\ d & 0 & b \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{C} \right\}$	$(x_1x_2, x_1{}^2, -x_2x_3, -x_1x_4)$
$\mathbb{C} \times A'$ *avec* $rang(A') = 2$	$(x_2^2 + x_3^2 + x_4^2\,,\, x_1x_2\,,\, -x_1x_3\,,\, -x_1x_4)$
A_*	$(x_1x_2\,,\, x_1{}^2\,,\, x_4{}^2 - x_2x_3\,,\, x_1x_4)$

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- In this way the classification of $\mathbf{Bir}_{2,2}(\mathbb{P}^r)$ with $r \leq 7$ easily follows.
- Semple (1929) describes *general elements* of $\mathbf{Bir}_{2,2}(\mathbb{P}^4)$.
- By geometrical methods Bruno and Verra reconsidered Semple's classification and generalized it to $\mathbb{P}^5$ with a description of general elements.

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In particular, if $X = X(r+1, n, \delta) \subset \mathbb{P}^N$, then

$$\deg(X) \leq \frac{\delta^{r+1}}{(n-1)^r} . \tag{2}$$