Arithmetic of singular Enriques surfaces

(joint with Klaus Hulek)

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[Non-trivial: showing that the lines generate $\mathrm{NS}(X)$.]

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Torelli: $X \cong Y \iff T(X) \cong T(Y)$

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Subtle point for 2.: Kummer surfaces have

$$T(\mathrm{Km}(A)) = T(A)(2),$$

so Kummer surfaces do not suffice to prove surjectivity.

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2. Shioda-Inose surface for $E_i \times E_{4i}$.

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Consequence: singular abelian surface A defined over $H(d)$, modular ($\rightsquigarrow \psi^2$)

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   (and converse by Elkies–S '08)

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Proof: geometric in nature, combining Shioda–Inose structure and Kummer sandwich structure

Shioda–Inose structure revisited

Shioda–Inose structure relies on elliptic fibrations (over $H(d)$):

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f is a quadratic base change ramifying at the I_0^* fibers (replaced by smooth fibers F_0, F_∞ in X)

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Conclusion: $\mathrm{MW}(X) \begin{cases} \text{invariant for } \jmath^* \\ \text{anti-invariant for } \imath^* \end{cases}$

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Obtain **involution of base change type** $\tau = \imath \circ t_P$ on X, since here $\imath \circ t_P = t_{-P} \circ \imath$

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Answer lattice theoretically in terms of intersection behaviour of section on $\mathrm{Km}(A)$ with I_0^* fibers

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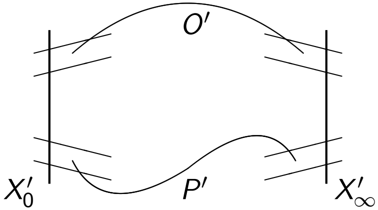

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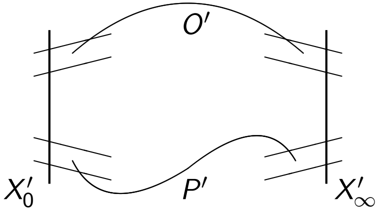

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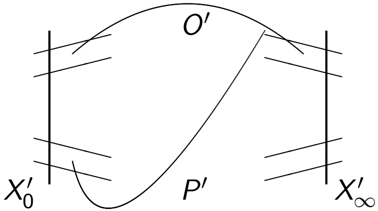

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$\implies P$ defined over quadratic extension of $H(d)$, quadratic twist of X has $P/H(d)$

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Then Y has a model over $H(d)$.

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- If a Galois element σ leaves $\mathrm{NS}(X)$ invariant, then τ and τ^σ induce the same action on $T(X)$ and on $\mathrm{NS}(X)$, so $\tau = \tau^\sigma$ by Torelli.

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$A = E \times E' \Longrightarrow$ projections induce elliptic fibrations

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(and be less sloppy with notation $\mathrm{MWL}(\mathrm{Km}(A))$ which so far always referred to the elliptic fibration in the Shioda–Inose structure)

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Proof: Fiber components, 2-torsion defined over $H(4d)$ □

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\begin{array}{ccccccc}
 & & & & & & \mathrm{Km}(A) \\
 & & & & & \swarrow & \downarrow \pi' \\
A & & & & X & & \mathbb{P}^1 \\
 & \searrow & & \swarrow & \downarrow & \swarrow & \\
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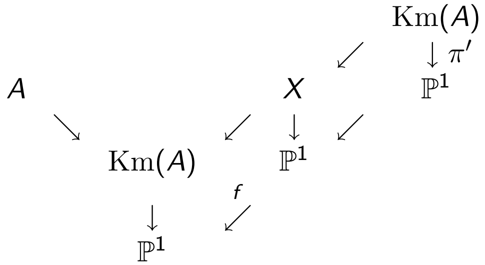

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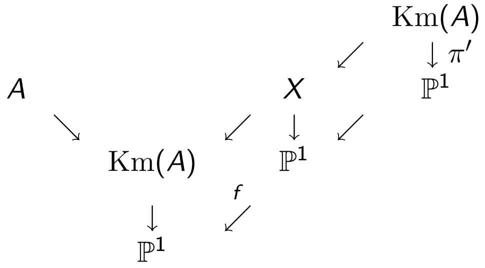

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Idea: compare image M with $\mathrm{MWL}(\mathrm{Km}(A), \pi)$

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(Since endowing π' with a section is achieved by fixing a base point of the cubic pencil $\{f(t)y^2 = g(x)\}$.)

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Theorem. X has a model with $\mathrm{NS}(X)$ over $H(d)$.

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Example: $X=\mathrm{Km}(E_\varrho^2)$, $\varrho^2+\varrho+1=0$: Shioda–Inose construction for $j=0, j'=60^3/4 \rightsquigarrow B=2^5\cdot 3\cdot 11\sqrt{-1}$.

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Thank you

&

all the best wishes to
Alessandro, Ciro and Fabrizio!