Abel-Jacobi map, integral Hodge classes, and decomposition of the diagonal

Claire Voisin

CNRS and Institut de Mathématiques de Jussieu

Levico Terme, September 2010

X=smooth projective complex variety.

- Assume $CH_0(X) = \mathbb{Z}$ (equivalently $CH_0(X)_{\mathbb{Q}} = \mathbb{Q}$).

Theorem

(Bloch-Srinivas) For some integer $N > 0$, one has an equality (Chow decomposition of the diagonal)

$$N\Delta_X = Z + Z' \text{ in } CH^n(X \times X),\, n = dim\, X$$

with Z' supported on $X \times pt$, Z supported on $D \times X$, for some $D \nsubseteq X$.

In particular, one has then such a decomposition at the level of cohomology classes, i.e. in $H^{2n}(X \times X, \mathbb{Z})$.

Definition

X has an *integral cohomological* decomposition of the diagonal if a decomposition as above holds in $H^{2n}(X \times X, \mathbb{Z})$, with $N = 1$.

Question *(Q0)*

For which X with trivial CH_0 group does there exist an integral cohomological decomposition of the diagonal?

More generally, study the following invariant $N(X)$ of X: $N(X)$ is the GCD of the integers N appearing in a cohomological decomposition of the diagonal as above. (Can also study the similar invariant defined using the Chow decomposition of the diagonal.)

Remark

This is a birational invariant of X. Indeed, under blow-up $\tau : Y \to X$, there is a decomposition

$$[\Delta_Y] = (\tau,\tau)^*[\Delta_X] + [\Delta_\tau]$$

where the cycle Δ_τ is supported over $E \times E$, E=exceptional divisor.

From now on, $dim\, X = 3$.
The Chow decomposition of the diagonal $N\Delta_X \equiv_{rat} Z + N(X \times pt)$ with Z supported over $D \times X$, $D \subsetneq X$ implies:

a) $CH^2(X)_{hom}/CH^2(X)_{alg}$ is of torsion (annihilated by N).

b) ϕ_X is surjective and $Ker\,(\phi_X : CH^2(X)_{hom} \to J(X))$ is of torsion (annihilated by N).
Here ϕ_X is the Abel-Jacobi map of X. $J(X)$ =Griffiths' intermediate Jacobian (an abelian variety in this case).

Much more is true :

Theorem (*Bloch, Bloch-Srinivas*)

Under the same assumptions:
a) $CH^2(X)_{hom}/CH^2(X)_{alg} = \{0\}$.
b) $Ker\,(\phi_X : CH^2(X)_{hom} \to J(X)) = \{0\}$.

(Uses Bloch-Ogus theory and Merkureev-Suslin theorem).

For such X, 1-cycles look very much like 0-cycles on curves. Namely ϕ_X induces an isomorphism: $CH^2(X)_{hom} \cong J(X)$.
Note $CH^2(X)_{hom}$ is an abstract group, a priori not a variety. $J(X)$ is a variety. The group morphism ϕ_X is algebraic in the following sense (this can be taken as an universal definition of $J(X)$):
For any smooth variety B, for any cod. 2 cycle $Z_B \subset B \times X$, s.t. Z_b is cohomologous to 0, $\forall b \in B$, the map

$$\phi_{Z_B} : B \to J(X), \, b \mapsto \phi_X(Z_b)$$

is a morphism of alg. varieties.

Question *(Q1)*

Same assumptions on X. Does there exist a cod. 2 cycle $Z_{J(X)} \subset J(X) \times X$, s.t. Z_t is cohomologous to 0, $t \in J(X)$ and

$$\phi_{Z_{J(X)}} : J(X) \to J(X), \quad t \mapsto \phi_X(Z_t)$$

is the identity of $J(X)$?

NB. For 0-cycles on curves, the analogous question has a positive answer (the universal divisor on $Pic^0(C) \times C$).

Remark

There is an integral Hodge class of degree 4 *on* $J(X) \times X$*, which corresponds to the isomorphism of Hodge structures* $H_1(J(X), \mathbb{Z}) \cong H^3(X, \mathbb{Z})/torsion$*. Thus (Q1) has an affirmative answer if the Hodge conjecture holds for degree* 4 *integral Hodge classes on* $J(X) \times X$*.*

Note: The Hodge conjecture does not hold in general for integral degree 4 Hodge classes (Atiyah-Hirzebruch, Kollár...), even on unirational varieties (Colliot-Thélène-Voisin 2010).

Remark

Answer to (Q1) is birationally invariant. More generally: the GCD of $deg\, f : B \to J(X)$*,* f *onto gen. finite, induced by a cycle* $Z \subset B \times X$*, i.e.* $f(b) = \phi_X(Z_b)$*, is a birational invariant of* X*.*

There are useful variants of the previous question :

Question *(Q2)*

Same assumptions on X. Does there exist a smooth projective variety B, and a cod. 2 cycle $Z_B \subset B \times X$, s.t. Z_b is cohomologous to 0, $b \in B$ and

$$\phi_{Z_B} : B \to J(X), \;\; b \mapsto \phi_X(Z_b)$$

is surjective with rationally connected general fibers?

(Compare with the case of zero cycles on curves: the Abel map

$$z \mapsto alb_C(z - z_0), \; C^{(n)} \to J(C)$$

is surjective with RC fibers for $n > g$).

Note: Positive answer to (Q1) $\Rightarrow$ Positive answer to (Q2). Take $B = J(X)$.

Proposition (Voisin 2010)

Assume (Q2) has an affirmative answer, and that there exists a 1*-cycle* $\Gamma \in CH_1(J)$ *such that* $\Gamma^{*g} = g!J(X)$, $g = dim\, J(X)$. *Then (Q1) also has an affirmative answer.*

Remark

As $dim\, X = 3$, $J(X)$ is a ppav. (Θ divisor given by the unimodular intersection pairing on $H^3(X, \mathbb{Z})/torsion$).

There is thus an integral Hodge class $\gamma = \frac{[\Theta]^{*(g-1)}}{(g-1)!}$ on $J(X)$, where $g = \dim J(X)$. It satisfies $\gamma^{*g} = g![J(X)]$, but is not known in general to be algebraic. This is known if $J(X)$ is a product of Jacobians of curves, for example if $g \leq 3$.

Proof of Proposition (sketch)

Assume for simplicity Γ is effective. By assumption, there exist a smooth projective variety B, and a cod. 2 cycle $Z \subset B \times X$, s.t. Z_b is cohomologous to 0, $b \in B$ and $\phi_{Z_B} : B \to J(X)$ is surjective with rationally connected general fibers.

• May assume by translating $\Gamma \subset J(X)$ that general fibers over Γ are rationally connected.

• The Graber-Harris-Starr theorem then says : there exists a lift $\gamma : \Gamma \to B$ of ϕ_B over Γ.

• Let $Z_\Gamma := (\gamma, Id_X)^* Z_B$. Then $\phi_\Gamma : \Gamma \to J(X),\ \gamma \mapsto \phi_X(Z_{\Gamma,\gamma})$ is the inclusion of Γ in $J(X)$.

• Z_Γ induces an obvious cod. 2 cycle $Z_\Gamma^{(g)} \subset \Gamma^{(g)} \times X$,

$$Z^{(g)}_{\Gamma^{(g)},\gamma_1+\ldots+\gamma_g} := Z_{\Gamma,\gamma_1} + \ldots + Z_{\Gamma,\gamma_g}.$$

• Now use the sum map, which is by assumption birational:

$$\mu : \Gamma^{(g)} \to J(X).$$

Let $Z_{J(X)} := (\mu, Id_X)_*(Z^{(g)}) \subset J(X) \times X$. Check that $\phi_{Z_{J(X)}} = Id_{J(X)}$.

Under our assumptions on X, any degree 4 class $\alpha \in H^4(X, \mathbb{Z})$ is Hodge. Then get a torsor $J(X)_\alpha$ in which the Deligne cycle class map $\phi_{X,D}$ on cod. 2 cycles of class α takes value. (Analogue of $Pic^d(C)$).
Concretely, for any smooth variety B, and cod 2 cycle $Z_B \subset B \times X$ s.t. Z_b is of class α, get a morphism

$$\phi_{Z_B} : B \to J(X)_\alpha,\, b \mapsto \phi_{X,D}(Z_{B,b}).$$

Question *(Q3)*

Does there exist a smooth projective variety B_α **canonically defined up to birational transformations**, *and a cod. 2 cycle $Z_\alpha \subset B_\alpha \times X$, s.t. Z_b is of class α, $b \in B_\alpha$ and*

$$\phi_{Z_\alpha} : B_\alpha \to J(X)_\alpha$$

is surjective with rationally connected general fibers?

Eg. It could be that, if X is rationally connected, for α sufficiently positive: (B_α=Hilbert scheme of rational curves of class α, Z_α=universal curve) works (question by Jason Starr).

A motivation for variant (Q3)

If B_α, Z_α are canonically defined, can put them in family.

- Let $\pi : \mathcal{X} \to \Gamma$, Γ= smooth proj. curve. $\mathcal{X}$ smooth proj. fourfold, π smooth over Γ_0.
- Let the generic fiber satisfy $H^2(\mathcal{O}_{\mathcal{X}_\eta}) = H^3(\mathcal{O}_{\mathcal{X}_\eta}) = 0$ (eg, $\mathcal{X}_t$ has CH_0 supported on a curve, $t \in \Gamma$ general).

This gives an algebraic family of abelian varieties $\mathcal{J} \to \Gamma_0$. For α section of $R^4\pi_*\mathbb{Z}$ over Γ_0, get twisted family $\mathcal{J}_\alpha \to \Gamma_0$.

- Assume $H^3(X_t, \mathbb{Z})$ has no torsion for any $t \in \Gamma_0$ and singular fibers of π have at most ordinary quadratic singularities.

Theorem (Colliot-Thélène-Voisin 2010)

Assume for any section α, there exists a family of codimension 2-cycles of class α in fibers of π:

$$B_\alpha \to \Gamma_0,\ Z_\alpha \subset B_\alpha \times_\Gamma \mathcal{X}$$

s.t. $\phi_{Z_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ is surjective with rationally connected general fibers. Then the Hodge conjecture is true for integral Hodge classes of degree 4 on $\mathcal{X}$.

• Use the theory of normal functions. A Hodge class β on $\mathcal{X}$ induces a section α of $R^4\pi_*\mathbb{Z}$ which has a lift ν_β to an algebraic section of $\mathcal{J}_\alpha$.
• By assumption, have B_α, $Z_\alpha \subset B_\alpha \times_\Gamma \mathcal{X}$, such that $\phi_{Z_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ is surjective with RC fibers. By Graber-Harris-Starr, ν_β has a lift to a section $\sigma : \Gamma \to B_\alpha$.
• Let $\mathcal{Z} := (\sigma, Id_{\mathcal{X}})^* Z_\alpha \subset \Gamma \times_\Gamma \mathcal{X} = \mathcal{X}$. The normal function associated to $\mathcal{Z}$ is equal to ν_β.
• As $H^3(X_t, \mathbb{Z})$ has no torsion, equality of normal functions implies that the degree 4 classes β and $[\mathcal{Z}]$ agree on $\mathcal{X}_0 := \pi^{-1}(\Gamma_0)$.
• The difference $[\mathcal{Z}] - \beta$ comes then from homology of singular fibers $H_4(X_{t_i}, \mathbb{Z})$.
• Assumptions $H^2(X_t, \mathcal{O}_{X_t}) = 0$ + singularities of X_{t_i} are at worst nodes $\Rightarrow$ this homology is generated by homology classes of 2-cycles on X_{t_i}. Thus $[\mathcal{Z}] - \beta$ is algebraic and so is β.

Theorem (*Voisin 2010*)

Let $\mathcal{X} \to \Gamma$ be a a smooth projective model of a cubic threefold in $\mathbb{P}^4_{\mathbb{C}(\Gamma)}$. Assume sing. fibers have at most ordinary quad. singularities. Then HC is true for integral Hodge classes of degree 4 on $\mathcal{X}$.

Check hypotheses: cubic threefolds X have trivial CH_0 (they are RC). No torsion in $H^3(X, \mathbb{Z})$ by Lefschetz. Note: $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ (degree).

Theorem

a) (Iliev-Markushevich 2002) The morphism induced by Abel-Jacobi map of X is surjective with RC fibers for the families B_4 of degree 4 rational curves and the family B_5 of degree 5 elliptic curves on X.
b) (Voisin 2010) The morphism induced by Abel-Jacobi map is surjective with RC fibers for the family B_6 of degree 6 elliptic curves on X.

$\Rightarrow$ existence of B_α, Z_α for all degrees. Indeed, use the cycle h^2 of degree 3 on fibers and its multiples to get then the result for all degrees.

Assume the existence of an integral cohomological decomposition

$$[\Delta_X] = [Z] + [Z']$$

with Z supported on $D \times X$, $D \subsetneq X$, Z' supported on $X \times pt$. This implies that $H^i(X, \mathcal{O}_X) = 0$, $i > 0$ by applying $[\Delta_X]^*$ to $H^i(X, \mathcal{O}_X)$, noticing that $[Z]^* = 0$ on $H^i(X, \mathcal{O}_X)$. (=Bloch-Srinivas' proof of Mumford's theorem).

Proposition (*Voisin 2010*)

Under this assumption, X satisfies :
a) $H^*(X, \mathbb{Z})$ has no torsion.
b) Positive answer to (Q1): there exists a codim 2 cycle $Z_J \subset J(X) \times X$ such that $\phi_{Z_J} : J(X) \to J(X)$ is $Id_{J(X)}$.
c) $H^4(X, \mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of algebraic cycles.

Sketch of proof.

Cycle classes act on integral cohomology and on Jacobians.
$[\Delta]^*$ acts as identity on integral cohomology and on Jacobians.
One has $[\Delta]^* = [Z]^*$ on $H^{*>0}(X, \mathbb{Z})$ and on $J(X)$.

• For a) in degree 3, get that $Id_{H^3(X,\mathbb{Z})}$ factors through $H^1(\widetilde{D}, \mathbb{Z})$. The later group has no torsion. Other degrees work similarly.

• For b) , get that $Id_{J(X)}$ factors through $Z^* : J(X) \to Pic^0(\widetilde{D})$. Here $j : \widetilde{D} \to X$ is a desing. of D. Z is lifted to a codim 2 cycle in $\widetilde{D} \times X$.
Let $\mathcal{D}$:= universal divisor on $Pic^0(\widetilde{D}) \times \widetilde{D}$.

• Let $Z_J = (Id_{J(X)}, j)_*((Z^*, Id_{\widetilde{D}})^*\mathcal{D}) \subset J(X) \times X$.
• Check that $\phi_{Z_J} = Id_{J(X)}$.

• For c), get for any $\alpha \in H^4(X, \mathbb{Z})$, by applying $[\Delta]^*$, that $\alpha = j_*([Z]^*\alpha)$, where Z is seen as a correspondence between $\widetilde{D}$ and X. But $[Z]^*\alpha$ is a degree 2 integral Hodge class on $\widetilde{D}$, hence algebraic by Lefschetz.

Partial converse

Assume X = smooth proj. threefold with $H^i(X, \mathcal{O}_X) = 0$, $i > 0$. Hence the Hodge structures on $H^2(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ are trivial. $J(X)$ is a ppav.

Theorem (*Voisin 2010*)

Assume
i) $H^*(X, \mathbb{Z})$ *has no torsion.*
ii) The intermediate Jacobian $J(X)$ *has a* 1*-cycle of class* $\frac{[\Theta]^{g-1}}{(g-1)!}$.
iii) question (Q1) has affirmative answer for X*, i.e. there is a codim 2 cycle* $Z_J \subset J(X) \times X$ *st.* Z_t *cohomologous to* 0 *on* X *for all* t*, with* $\phi_{Z_J} = Id : J(X) \to J(X)$.
iv) $H^4(X, \mathbb{Z})$ *is algebraic.*
Then X *admits an integral cohomological decomposition of the diagonal.*

Remark

When X is a uniruled threefold with $H^2(X, \mathcal{O}_X) = 0$, it is known (Voisin 2006) that $H^4(X, \mathbb{Z})$ is algebraic, i.e. iv) holds.

• For a topological manifold with no torsion in $H^*(X,\mathbb{Z})$, there is a Künneth decomposition of cohomology of $X \times X$. Thus $[\Delta_X] = \delta_{6,0} + \delta_{5,1} + \delta_{4,2} + \delta_{3,3} + \delta_{2,4} + \delta_{1,5} + \delta_{0,6}$.
• As $H^1(X,\mathcal{O}_X) = 0$, $\delta_{5,1} = \delta_{1,5} = 0$.
• As $H^2(X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})$ are generated by cycle classes, both $\delta_{4,2}$ and $\delta_{2,4}$ are classes of algebraic cycles supported over $D \subsetneq X$.
It only remains to construct a cycle $Z_3 \subset X \times X$ s.t. Z_3 is supported over some $D \subsetneq X$ and $[Z_3]$ acts as identity on $H^3(X,\mathbb{Z})$.
• There is a 1-cycle Γ in $J(X)$ with class $[\Gamma] = \frac{\Theta^{g-1}}{(g-1)!}$. Assume for simplicity Γ effective (so $J(X)$ is a Jacobian).
• There is $Z_J \subset J(X) \times X$ codim 2 cycle st. $\phi_{Z_J} = Id : J(X) \to J(X)$. Let $Z_\Gamma := Z_{J|\Gamma \times X}$.
• Let $Z_3 := Z_\Gamma \circ {}^tZ_\Gamma$. Z_3 is supported over a surface in X, as ${}^tZ_\Gamma$. Check that $[Z_3]$ acts as identity on $H^3(X,\mathbb{Z})$.