# On maximal curves over finite fields of small order

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# Notation and terminology

- $\mathcal{X} \subseteq \mathbb{P}^r(\bar{\mathbb{F}}_\ell)$ projective, geometrically irreducible, non-singular algebraic curve, defined over $\mathbb{F}_\ell$
- $g$ genus of $\mathcal{X}$
- If $f(X,Y)=0$ is birationally equivalent to $\mathcal{X}$, then

$$\mathcal{X} : f(X,Y)=0$$

  $\mathcal{X}$ is said to be the non-singular model of $f(X,Y)=0$
- $\mathcal{X}(\mathbb{F}_\ell) = \mathcal{X} \cap PG(r,\ell)$

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# Maximal Curves

**Theorem (Hasse-Weil, 1948)**

$$|\mathcal{X}(\mathbb{F}_\ell)| \leq \ell + 1 + 2g\sqrt{\ell}.$$

**Definition**

*$\mathcal{X}$ is $\mathbb{F}_\ell$-maximal (or simply maximal) if the number $|\mathcal{X}(\mathbb{F}_\ell)|$ of its $\mathbb{F}_\ell$-rational points attains the equality in the Hasse-Weil bound.*

- $\ell$ square, $\ell = q^2$ ($q$ power of a prime).

**Example: Hermitian curve**

$$\mathcal{H}_2 : X_2^{q+1} = X_1^q X_0 + X_1 X_0^q$$

- $g = \frac{1}{2}q(q-1)$, $\quad |\mathcal{H}_2(\mathbb{F}_{q^2})| = q^3 + 1$

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# Relevance of maximal curves in the construction of good AG-Codes

The quality of a linear code is measured by the following parameters:

- *Information Rate* $R$
- *Relative Distance* $\delta$

- Theorem (Singleton): $R + \delta \leq 1 + 1/n$
- A code is said to be "*good*" when $R + \delta$ is near to 1.

If $C$ is an AG-code built from a non-singular curve $\mathcal{X}$, of genus $g$, defined over $\mathbb{F}_\ell$, then

$$R + \delta \geq 1 - \frac{g-1}{|\mathcal{X}(\mathbb{F}_\ell)|}.$$

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# Natural Embedding Theorem

**Hermitian Variety of $\mathbb{P}^m(\overline{\mathbb{F}}_{q^2})$**

$$\mathcal{H}_m : X_2^{q+1} + X_3^{q+1} + \cdots + X_m^{q+1} = X_1^q X_0 + X_1 X_0^q$$

**Theorem (G. Korchmáros - F. Torres, 2001)**

*Up to isomorphisms, $\mathbb{F}_{q^2}$-maximal curves are:*

- *non-singular irreducible curves, of degree $q+1$,*
- *contained in some non-degenerate $\mathcal{H}_m$;*
- *thus $2 \leq m \leq r$.*

The integer $r$ is the geometrical *Frobenius dimension* of the curve.

- $r = 2 \Rightarrow m = 2 \Rightarrow \mathcal{X} \cong \mathcal{H}_2$.
- $r = 3 \Rightarrow m = 3$.

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# Classification results

- largest genus: $g_1 = \frac{1}{2}q(q-1)$ (Ihara, 1981)
  If $g(\mathcal{X}) = \frac{1}{2}q(q-1)$, then $\mathcal{X} \cong \mathcal{H}_2$ (Rück-Stichtenoth, 1994).
- second largest genus: $g_2 = \left\lfloor \frac{1}{4}(q-1)^2 \right\rfloor$ (Fuhrmann-Torres, 1996).
  If $g(\mathcal{X}) = \left\lfloor \frac{1}{4}(q-1)^2 \right\rfloor$, then
  - for $q$ odd, $\mathcal{X}$ is $\mathbb{F}_{q^2}$-birationally equivalent to the curve of affine equation
    $$Y^{\frac{q+1}{2}} = X^q + X$$
    (Fuhrmann-Garcia-Torres, 1997);
  - for $q \geq 4$ even, $\mathcal{X}$ is $\mathbb{F}_{q^2}$-birationally equivalent to the curve of affine equation
    $$Y^{q+1} = X + X^2 + X^4 + \ldots + X^{\frac{q}{4}} + X^{\frac{q}{2}}$$
    (Abdón-Torres 1999 for $q = 4$, Korchmáros-Torres 2002 for $q > 4$).
- third largest genus: $g_3 = \left\lfloor \frac{1}{6}(q^2 - q + 4) \right\rfloor$ (Korchmáros-Torres, for $q \geq 7$)

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**Theorem (S. Fanali - M. Giulietti - I.P., 2012)**

*Up to birational equivalence, the curve $\mathcal{Y}(7)$ of affine equation*

$$Y^7 - YX^4 + \omega X^2 = 0, \text{ con } \omega^8 = -1.$$

*is the only $\mathbb{F}_{49}$-maximal curve of genus 7.*

# Sketch of the proof

Let $\mathcal{X}$ be an $\mathbb{F}_{49}$-maximal curve, of genus 7.

- $\begin{cases} q = 7 \\ g = 7 \end{cases} \Rightarrow r = 3.$

Determine all the possible model plane of $\mathcal{X}$, using:

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## Curves with Frobenius dimension 3

- $H(P) = \{0, \rho(P), q, q+1, \dots\}$

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*If $\mathcal{X}$ is an $\mathbb{F}_{q^2}$-maximal curve with Frobenius dimension 3 and $P \in \mathcal{X}(\mathbb{F}_{q^2})$, then $\mathcal{X}$ is birationally equivalent over $\mathbb{F}_{q^2}$ to a plane curve with affine equation*

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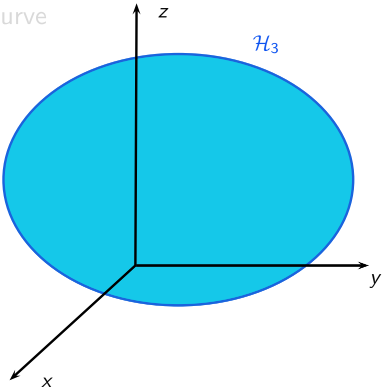

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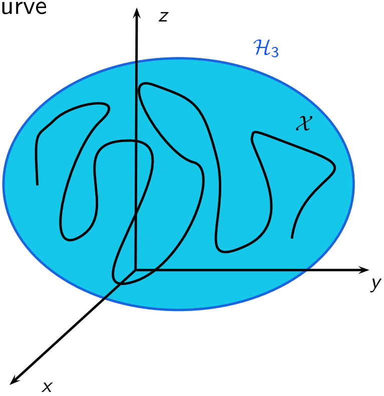

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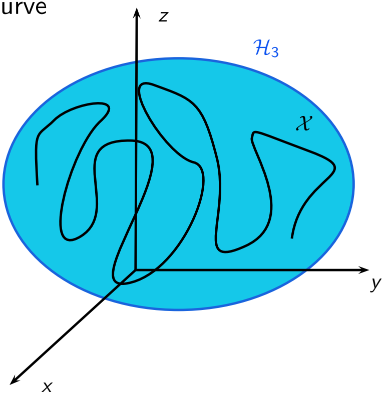

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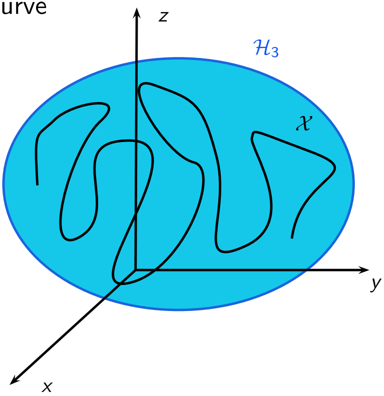

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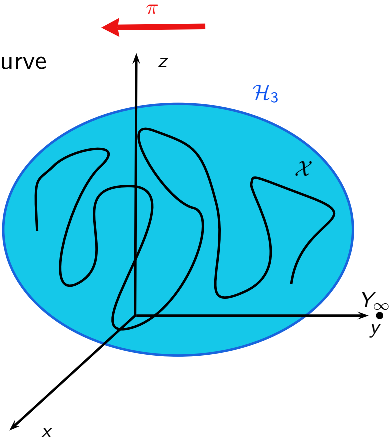


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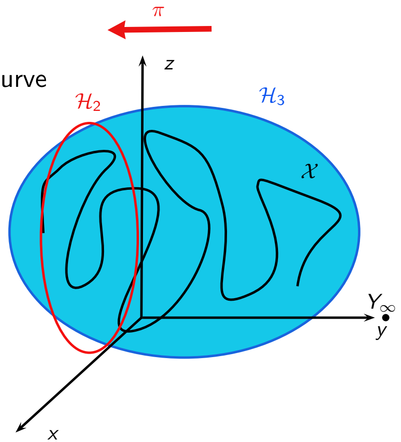


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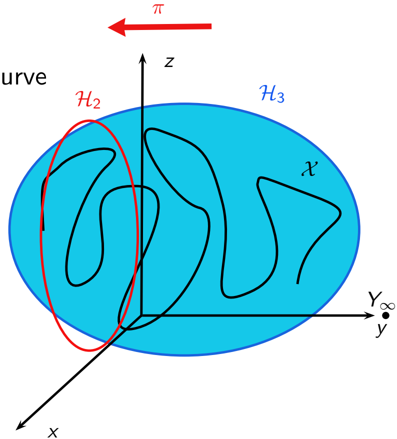


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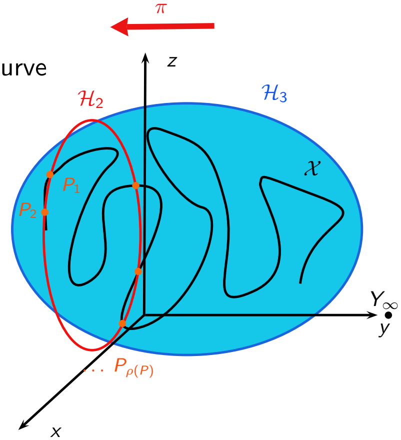

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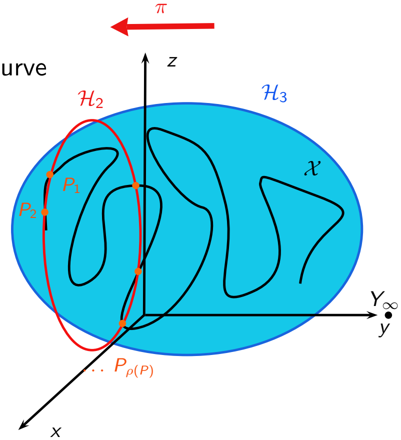
$$Z^{q+1} + Y^{q+1} = X^q T + X T^q, \qquad \mathcal{H}_3 \subseteq \mathbb{P}^3(\bar{\mathbb{F}}_{q^2})$$

$\overset{N.E.T}{\Rightarrow} \begin{cases} \mathcal{X} \text{ is an irreducible, non-singular curve} \\ \deg(\mathcal{X}) = q+1 \\ \mathcal{X} \subseteq \mathcal{H}_3 \end{cases}$

Let $\pi$ be the canonical projection from $Y_\infty$ to the $xz$-plane; then:

- $\pi(\mathcal{H}_3) = \mathcal{H}_2$
- $\pi(\mathcal{X}) \subseteq \mathcal{H}_2$
- $\mathcal{X} \cap \{Y = 0\} \subseteq \mathcal{H}_2 \rightarrow$ theese are the $P_1, \ldots, P_{\rho(P)}$ points of the Theorem!



# Curves with Frobenius dimension 3

- $H(P) = \{0, \rho(P), q, q+1, \ldots\}$

**Theorem (S. Fanali - M .Giulietti - I.P., 2012)**

*If $\mathcal{X}$ is an $\mathbb{F}_{q^2}$-maximal curve with Frobenius dimension 3 and $P \in \mathcal{X}(\mathbb{F}_{q^2})$, then $\mathcal{X}$ is birationally equivalent over $\mathbb{F}_{q^2}$ to a plane curve with affine equation*

$$Z^{q+1} = X^q + X + \lambda\xi(X, Z),$$

*where*

- $\lambda \in \mathbb{F}^*_{q^2}$
- $\xi(X, Z)$ *is a polynomial of degree* $\rho(P)$
- $\xi(X, Z) = \prod_{i=1,\ldots,\rho(P)} T_i(X, Z)$
- $T_i(X, Z) = 0$ *are tangents of* $\mathcal{H}_2$ *at* $\rho(P)$ *not necessarily distinct affine points* $P_1, \ldots, P_{\rho(P)}$.

# Estimate the number of curves to be tested

Let $N$ be the number of curves to be tested.

- We have

$$N = \underbrace{(q^2-1)}_{\substack{\downarrow \\ \lambda \in \mathbb{F}_{q^2}^*}} \cdot \underbrace{(q^3)^{\rho(P)}}_{\substack{\downarrow \\ \text{ways in which} \\ \xi(X,Z) \text{ can change}}} \approx q^{2+3\rho(P)}$$

- For $q=7$, since $j(P)>1$, we obtain

$$\rho(P) = q+1-j(P) \leq 6,$$

and so we have, in the worst case

$$N \approx 7^{20} (\approx 2^{56}) \longrightarrow \text{ not computationally manageable!}$$

- **Geometric remarks necessary to reduce $N$ and computation time for testing each curve.**

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# Classification of maximal curves defined over $\mathbb{F}_{49}$, of genus 7

Let $\mathcal{X}$ be a maximal curve defined over $\mathbb{F}_{49}$, of genus 7. We may assume that:

**i)** $\mathcal{X}$ is a curve of degree 8, lying on the Hermitian surface of $\mathbb{P}^3(\overline{\mathbb{F}}_{49})$ of equation $Z^8 + Y^8 = X^7T + XT^7$; $(\overset{\text{N.E.T.}}{\Rightarrow})$

**ii)** $P = (0:1:0:0) \in \mathcal{X}$;

**iii)** the osculating plane to $\mathcal{X}$ in $P$ has equation $T = 0$;

**iv)** the tangent line $\mathcal{X}$ in $P$ has equations $Y = 0,\ T = 0$;

**v)** the non-osculating tangent planes to $\mathcal{X}$ in $P$ are that of equation

$$Y - bT = 0, \text{ with } b \in \overline{\mathbb{F}}_{49}. \; (\overset{\text{iii)-iv)}}{\Rightarrow})$$

**Proposition**

*Let $j(P)$ be the multiplicity intersection of $\mathcal{X}$ in $P$ with an arbitrary non-osculating tangent plane to $\mathcal{X}$ in $P$. Then $j(P) = 2$ or $j(P) = 3$.*

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# Case $j(P) = 3$

- If $j(P) = 3 \overset{i)}{\Rightarrow}$ every non-osculating tangent plane to $\mathcal{X}$ in $P$ intersects the curve $\mathcal{X}$ in 5 not necessarily distinct affine points, counted with multiplicity.

**Proposition**

*There exists a non-osculating tangent plane $H$ to $\mathcal{X}$ in $P$ of equation*

$$Y - bT = 0, \text{ con } b \in \mathbb{F}_{49},$$

*in which the 5 affine points of $\mathcal{X} \cap H$ are not all distinct, (i.e. the multiplicity of intersection of the plane with $\mathcal{X}$ in one of theese points is greater than 1).*

- Up to projectivity we can assume that:
  1. $H : Y = 0$;
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## Frobenius morfism, defined over $\mathbb{F}_q$

$$\begin{array}{rccc} \Phi_q & : & \mathbb{P}^n(\overline{\mathbb{F}}_q) & \rightarrow & \mathbb{P}^n(\overline{\mathbb{F}}_q) \\ & & (X_0 : \ldots : X_n) & \mapsto & (X_0^q : \ldots : X_n^q) \end{array}$$

## Remark

*Acting with Frobenius morfism $\Phi_7$, defined over $\mathbb{F}_7$ it is possible to divide the $\mathbb{F}_{49}$-rational points of $\mathcal{H}_2$ in three disjoint orbits:*

- $\mathcal{A}_1 := \{Q \in \mathcal{H}_2(\mathbb{F}_{49}) \ : \ \Phi_7(Q) = Q\}$;
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$$Q \in \mathcal{A}_2 \Rightarrow \Phi_7(Q) \notin \mathcal{A}_2;$$

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$$Q \in \mathcal{A}_2 \Rightarrow \Phi_7(Q) \notin \mathcal{A}_2;$$

- $\mathcal{A}_3 := \{\Phi_7(Q) \mid Q \in \mathcal{A}_2\}$.

# $j(P) = 3$

We limited our search considering the following cases ($P_0$ and $P_1$ previously defined):

**A1)** $P_2, P_3 \in \mathcal{A}_1$;

**A2)** $P_2 \in \mathcal{A}_1$, $P_3 \in \mathcal{A}_2$;

**A3)** $P_2 \in \mathcal{A}_2$, $P_3 \in \mathcal{A}_3$;

**A4)** $P_2, P_3 \in \mathcal{A}_2$.

# Estimate the number of curves to be tested in the case $j(P) = 3$

- Number of curves to be tested: $N \approx 7^{2+3\rho(P)}$.
- If $j(P) = 3 \Rightarrow \rho(P) = 5$ and so $N \approx 7^{17} (\approx 2^{47})$.

Thanks to the previous geometric remarks, we have significantly reduced the value of $N$.

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# Other topics

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Thanks for your attention!