Random Network Codes and Algebraic Curves

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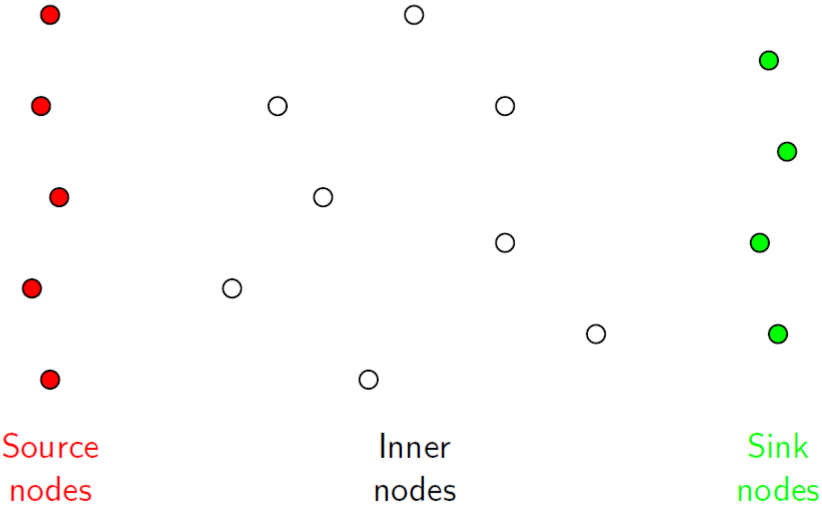
(joint work with Daniele Bartoli and Massimo Giulietti)

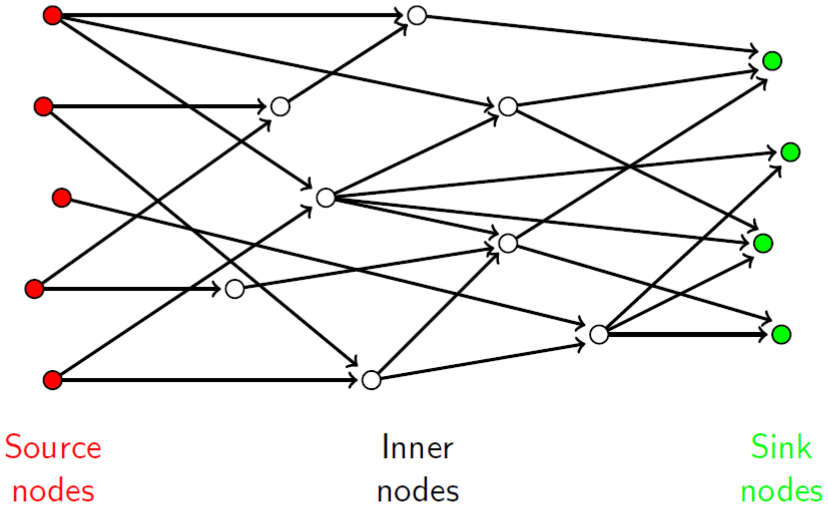
University of Trento

17 December 2015

Source nodes


Source nodes
Sink nodes


Source nodes
Inner nodes
Sink nodes



Source
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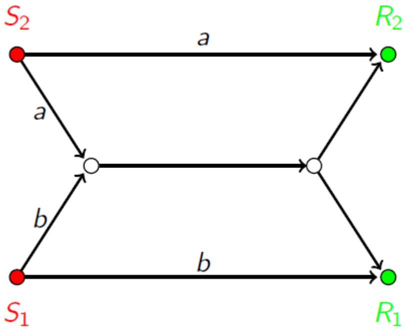
Butterfly Network



S_1 sends only b
S_2 sends only a

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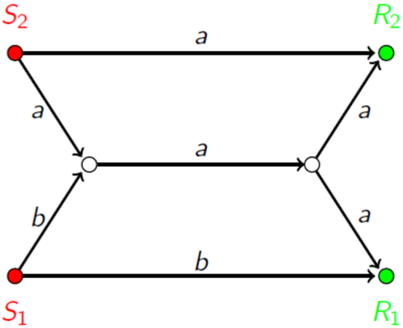

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R_1 receives a AND b

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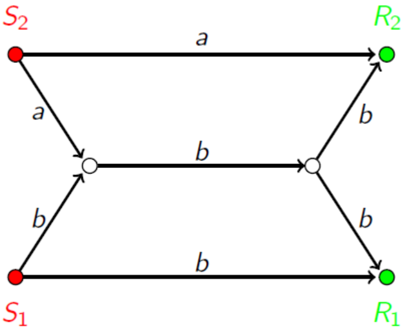

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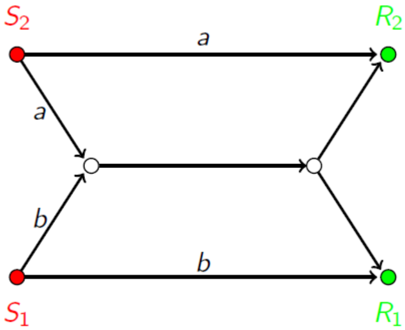

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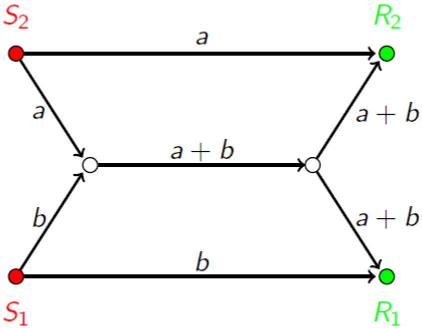

S_1 sends only b
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R_1 receives b AND $a+b \Rightarrow a$
R_2 receives a AND $a+b \Rightarrow b$

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- The inner nodes make a random linear $\mathbb{F}_q$-combination of the vectors received.
- The vector space that arrives to the sinks is the same sent from the sources.

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The set of all $\mathbb{F}_q$-linear subspaces of $\mathbb{F}_q^N$ with fixed dimension $\ell \leq N$ is named **Grassmannian** $\mathcal{G}_q(N, \ell)$.

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Definition [Kötter, Kschischang. 2008]

We define the subspace distance $d : \mathcal{G}_q(N, \ell) \times \mathcal{G}_q(N, \ell) \to \mathbb{Z}_+$ by

$$d(U, V) := \dim(U + V) - \dim(U \cap V).$$

From Grassman's formula we have that

$$\begin{aligned} d(U, V) &= \dim(U) + \dim(V) - 2\dim(U \cap V) \\ &= 2\ell - 2\dim(U \cap V) \end{aligned}$$

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A **Network Code** is a set $C \subseteq \mathcal{G}_q(n,\ell)$. C is a $[N, \ell, \log_q |C|, D(C)]$-code where:

- N $\rightarrow$ lenght
- $\log_q |C|$ $\rightarrow$ dimension
- $D(C) := \min_{\substack{U, V \in C \\ U \neq V}} d(U,V)$ $\rightarrow$ minimum distance

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Definition

Given a $[N, \log_q |C|, D(C)]$-code

- $\lambda = \frac{\ell}{N} \rightarrow$ normalized weight
- $R = \frac{\log_q(|C|)}{N\ell} \rightarrow$ rate
- $\delta = \frac{D(C)}{2\ell} \rightarrow$ normalized minimum distance

Riemann-Roch Spaces

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Let W be a divisor with $\deg W = 2g - 2$ and D a divisor, then $\dim_{\mathbb{F}_q} \mathcal{L}(D) = \deg D + 1 - g + \dim_{\mathbb{F}_q} \mathcal{L}(W - D)$.

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Corollary

If $\deg D > 2g - 2$ then $\dim_{\mathbb{F}_q} \mathcal{L}(D) = \deg D + 1 - g$.

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Theorem (Hansen, 2015)

If $ks > 2g - 2$

$$\mathcal{H}_{k,s} \text{ is a } \left[\underbrace{kn+1-g}_{N}, \underbrace{ks+1-g}_{\ell}, \log_q \binom{n}{s}, D(\mathcal{H}_{k,s})\right] \text{-code}$$

$$D(\mathcal{H}_{k,s}) = \begin{cases} 2k, & s > 1; \\ 2(k+1-g), & s = 1. \end{cases}$$

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$S \subset \mathcal{X}(\mathbb{F}_q), |S| = s$ $\Rightarrow$

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Family $\mathcal{B}_{k,s,w}$

$\mathcal{A}_{k,s}$		$\mathcal{B}_{k,s,w}$
S multiset		S multiset
$Supp(S) \subset \mathcal{X}(\mathbb{F}_q)$	$\Rightarrow$	$Supp(S) \subset \mathcal{X}(\mathbb{F}_q)$
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Theorem

If $k > 2g - 2$ and $0 < w < s$

$$\boxed{\mathcal{B}_{k,s,w} \text{ is a } [\underbrace{nkw + 1 - g}_{N}, \underbrace{ks + 1 - g}_{\ell}, \log_q |\mathcal{B}_{k,s,w}|, 2k]\text{-code}}$$

$$|\mathcal{B}_{k,s,w}| = \sum_{i=0}^{t} (-1)^i \binom{n}{i} \binom{s - i(w+1) + n - 1}{n-1}$$

$$t = \min\left(n, \left\lfloor \frac{s}{b+1} \right\rfloor\right)$$

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$|S| = s,\ m_P \in \{0, \ldots, w\}$

$\Rightarrow$

S multiset
$Supp(S) \subset \mathcal{X}(\mathbb{F}_q)$
$\sum_{P \in \mathcal{X}(\mathbb{F}_q)} m_P = s$
$m_P \in \{\underbrace{s - w(n.1)}_{\overline{w}}, \ldots, w\}$

Construction of the code

- $k > 0,\ s > 0,\ 0 < w \le s$ fixed integers.
- $\mathcal{W} = \mathcal{L}(kwD)$ ambient space.

$$\mathcal{C}_{k,s,w} := \left\{ V = \mathcal{L}\left(k \sum_{P \in \mathcal{X}(\mathbb{F}_q)} m_P P \right) \subset \mathcal{W} \;\middle|\; \begin{array}{l} \sum_{P \in \mathcal{X}(\mathbb{F}_q)} m_P = s, \\ m_P \in \{\overline{w}, \ldots w\} \end{array} \right\}$$

Theorem

If $k > 2g - 2$ and $0 < w < s$

$$\mathcal{C}_{k,s,w} \text{ is a } [\underbrace{nkw + 1 - g, ks + 1 - g}_{N}, \underbrace{ks + 1 - g}_{\ell}, \log_q |\mathcal{C}_{k,s,w}|]_q\text{-code}$$

$$|\mathcal{C}_{k,s,w}| = \sum_{i=0}^{t} (-1)^i \binom{n}{i} \binom{(nw - s + 1)(n - i - 1)}{n - 1}$$

$$t = \left\lfloor \frac{(nw - s)(n - 1)}{nw - s + 1} \right\rfloor$$

Normalized weight, rate, and normalized minimum distance

	Normalized weight	Rate	Normalized minimum distance
$\mathcal{H}_{k,s}$	$\frac{ks+1-g}{nk+1-g}$	$\frac{\log_q \binom{n}{s}}{(nk+1-g)(ks+1-g)}$	$\frac{1}{s+\frac{1-g}{k}}$
$\mathcal{A}_{k,s}$	$\frac{ks+1-g}{nks+1-g}$	$\frac{\log_q \binom{n+s-1}{s}}{(nks+1-g)(ks+1-g)}$	$\frac{1}{s+\frac{1-g}{k}}$
$\mathcal{B}_{k,s,w}$	$\frac{ks+1-g}{nkw+1-g}$	$\frac{\log_q \lvert\mathcal{B}_{k,s,w}\rvert}{(nkw+1-g)(ks+1-g)}$	$\frac{1}{s+\frac{1-g}{k}}$
$\mathcal{C}_{k,s,w}$	$\frac{ks+1-g}{nkw+1-g}$	$\frac{\log_q \lvert\mathcal{C}_{k,s,w}\rvert}{(nkw+1-g)(ks+1-g)}$	$\frac{1}{s+\frac{1-g}{k}}$

- In the new constructions s can be larger than n.

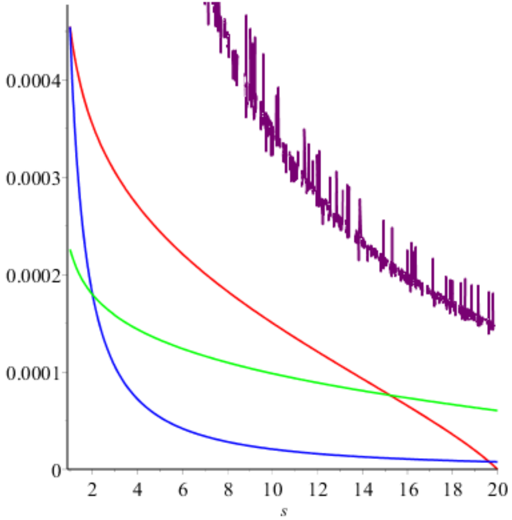
- In the new constructions s can be larger than n.
- We do not mind if n is small, so we are not constricted to choose only curves with many $\mathbb{F}_q$-rational points.

Rates of $\mathcal{H}_{k,s}$, $\mathcal{A}_{k,s}$, $\mathcal{B}_{k,s,w}$, $\mathcal{C}_{k,s,w}$ for $q = 16,\ 8 \leq n \leq 14,\ 1 \leq s < n,\ w = 3,\ k = 5$

(n,s)	$\mathcal{H}_{k,s}$	$\mathcal{A}_{k,s}$	$\mathcal{B}_{k,s,w}$	$\mathcal{C}_{k,s,w}$	(n,s)	$\mathcal{H}_{k,s}$	$\mathcal{A}_{k,s}$	$\mathcal{B}_{k,s,w}$	$\mathcal{C}_{k,s,w}$
(8, 1)	0.003750	0.003750	0.001250	0.008732	(12, 2)	0.002518	0.001309	0.000873	0.004617
(8, 2)	0.003005	0.001616	0.001077	0.004286	(12, 3)	0.002161	0.000788	0.000788	0.003040
(8, 3)	0.002420	0.000959	0.000959	0.002802	(12, 4)	0.001865	0.000542	0.000722	0.002251
(8, 4)	0.001915	0.000654	0.000868	0.002058	(12, 5)	0.001605	0.000403	0.000669	0.001778
(8, 5)	0.001452	0.000481	0.000792	0.001611	(12, 6)	0.001368	0.000315	0.000624	0.001461
(8, 6)	0.001002	0.000373	0.000728	0.001311	(12, 7)	0.001146	0.000254	0.000585	0.001234
(9, 1)	0.003522	0.003522	0.001174	0.008931	(12, 8)	0.000932	0.000211	0.000551	0.001064
(9, 2)	0.002872	0.001526	0.001017	0.004394	(12, 9)	0.000720	0.000179	0.000519	0.000931
(9, 3)	0.002368	0.000909	0.000909	0.002880	(13, 1)	0.002846	0.002846	0.000949	0.009440
(9, 4)	0.001938	0.000622	0.000826	0.002121	(13, 2)	0.002417	0.001251	0.000834	0.004670
(9, 5)	0.001551	0.000459	0.000758	0.001665	(13, 3)	0.002092	0.000755	0.000755	0.003079
(9, 6)	0.001184	0.000357	0.000700	0.001360	(13, 4)	0.001823	0.000521	0.000694	0.002282
(10, 1)	0.003322	0.003322	0.001107	0.009093	(13, 5)	0.001589	0.000388	0.000644	0.001804
(10, 2)	0.002746	0.001445	0.000964	0.004482	(13, 6)	0.001378	0.000303	0.000602	0.001485
(10, 3)	0.002302	0.000865	0.000865	0.002943	(13, 7)	0.001181	0.000245	0.000566	0.001256
(10, 4)	0.001929	0.000593	0.000788	0.002173	(13, 8)	0.000993	0.000204	0.000533	0.001085
(10, 5)	0.001595	0.000439	0.000726	0.001710	(13, 9)	0.000810	0.000173	0.000505	0.000950
(10, 6)	0.001286	0.000341	0.000673	0.001400	(13, 10)	0.000628	0.000148	0.000478	0.000843
(10, 7)	0.000987	0.000275	0.000627	0.001178	(14, 1)	0.002720	0.002720	0.000907	0.009486
(10, 8)	0.000686	0.000228	0.000586	0.001011	(14, 2)	0.002324	0.001199	0.000799	0.004705
(11, 1)	0.003145	0.003145	0.001048	0.009229	(14, 3)	0.002026	0.000725	0.000725	0.003102
(11, 2)	0.002628	0.001374	0.000916	0.004555	(14, 4)	0.001780	0.000501	0.000667	0.002307
(11, 3)	0.002232	0.000824	0.000824	0.002996	(14, 5)	0.001567	0.000373	0.000621	0.001827
(11, 4)	0.001901	0.000566	0.000754	0.002215	(14, 6)	0.001375	0.000292	0.000581	0.001511
(11, 5)	0.001609	0.000420	0.000697	0.001746	(14, 7)	0.001198	0.000237	0.000547	0.001252
(11, 6)	0.001341	0.000327	0.000648	0.001433	(14, 8)	0.001031	0.000197	0.000517	0.001106
(11, 7)	0.001087	0.000264	0.000606	0.001209	(14, 9)	0.000870	0.000167	0.000490	0.000974
(11, 8)	0.000837	0.000219	0.000568	0.001040	(14, 10)	0.000712	0.000144	0.000466	0.000865
(12, 1)	0.002987	0.002987	0.000996	0.009343	(14, 11)	0.000552	0.000125	0.000443	0.000768

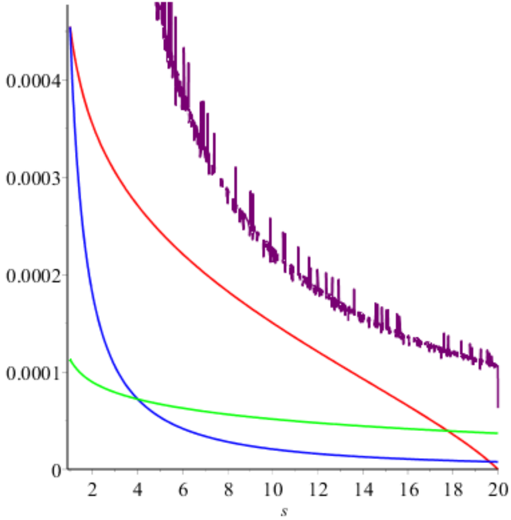
$\mathcal{H}_{k,s}$ $\mathcal{A}_{k,s}$ $\mathcal{B}_{k,s,w}$ $\mathcal{C}_{k,s,w}$

q=64 g=3 k=10 n=20 w=2



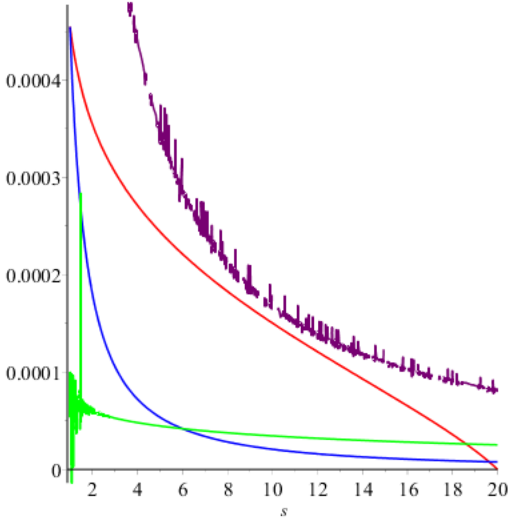
$\mathcal{H}_{k,s}$ $\mathcal{A}_{k,s}$ $\mathcal{B}_{k,s,w}$ $\mathcal{C}_{k,s,w}$

q=64 g=3 k=10 n=20 w=4



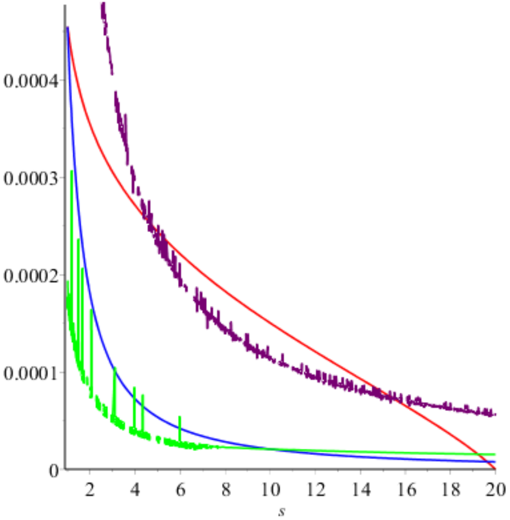
$\mathcal{H}_{k,s}$ $\mathcal{A}_{k,s}$ $\mathcal{B}_{k,s,w}$ $\mathcal{C}_{k,s,w}$

q=64 g=3 k=10 n=20 w=6



$\mathcal{H}_{k,s}$ $\mathcal{A}_{k,s}$ $\mathcal{B}_{k,s,w}$ $\mathcal{C}_{k,s,w}$

q=64 g=3 k=10 n=20 w=10



THANK YOU FOR YOUR ATTENTION