# Vectorial Boolean Functions in even dimension

Irene Villa

Università degli Studi di Trento


ALICE

BOB

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Communicate a secret message

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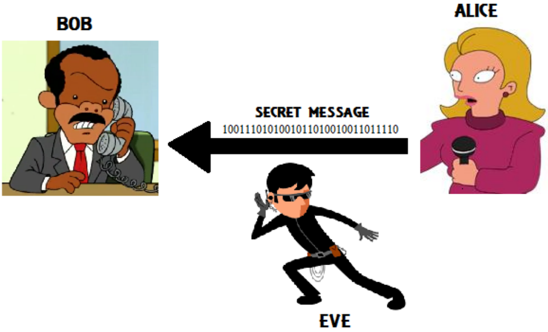

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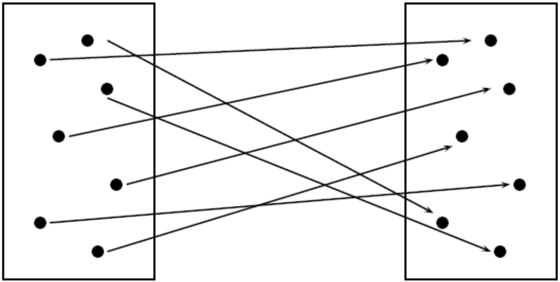
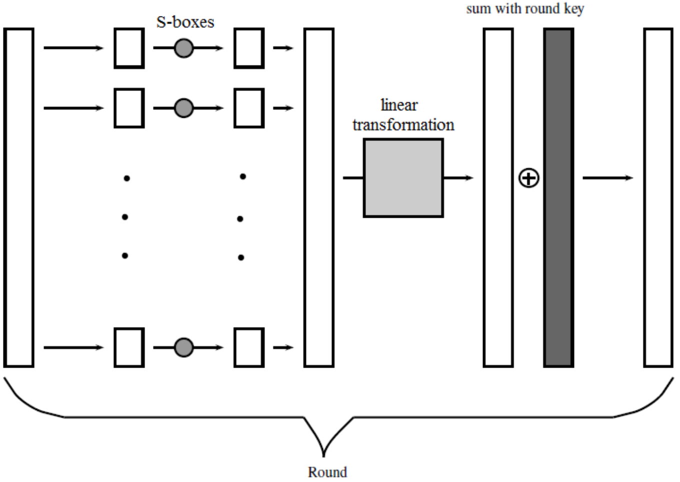


# Block ciphers



Example of translation based cipher



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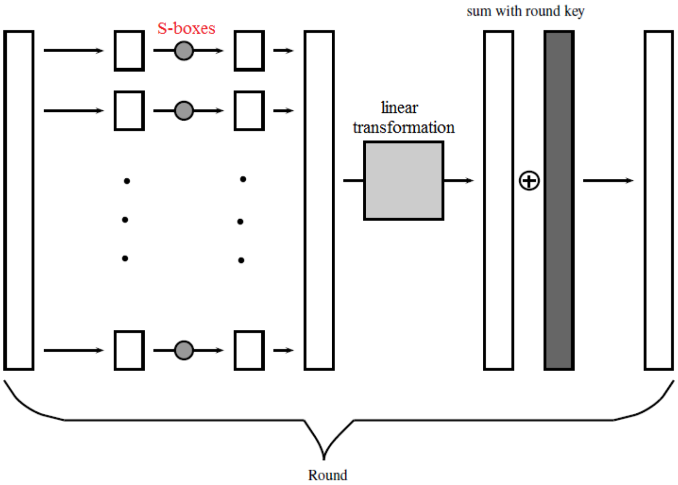

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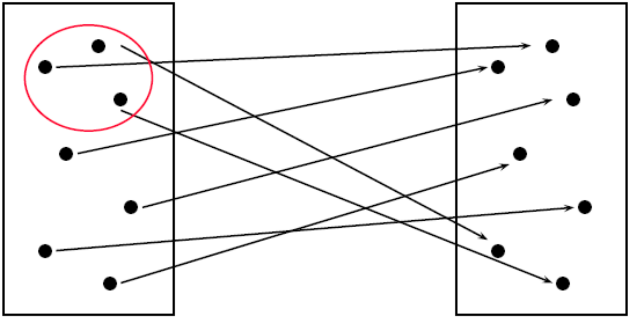

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## Proposition

Let $F : \mathbb{F}^n \to \mathbb{F}^n$ be an APN permutation. Then $F$ is AC iif $\hat{n} = 0$.

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Extend the results of J. Seberry, X.-M. Zhang, and Y. Zheng (1994) and K. Nyberg (1995).

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## Proposition

Let $F$ be a cubic APN permutation with $n$ even. Then $\exists \; \lambda$ s.t. $|\{a : \mathcal{F}(D_a f_\lambda) = 0\}| \geq 2^n - 2^{n-2} - 1$.

## Proposition

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**Theorem**

No 4-bit permutation can be APN.

Thank you for your attention