$$\ddot{x} + \omega^2 x = 0$$

$$z = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad \dot{z} = A\, z$$

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

$$A^2 = -\omega^2\, I, \quad A^{2k} = (-1)^k \omega^{2k}\, I$$

$$A^{2k+1} = (-1)^k \omega^{2k}\, A$$

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n\, A^n}{n!} = \Big(\sum_{k=0}^{\infty} (-1)^k \omega^{2k} \frac{t^{2k}}{(2k)!} \Big)\, I + \Big(\sum_{k=0}^{\infty} (-1)^k \omega^{2k} \frac{t^{2k+1}}{(2k+1)!} \Big)\, A$$

$$e^{tA} = \Big(\cos(\omega t) \Big)\, I + \Big(\frac{1}{\omega}\, \sin(\omega t) \Big)\, A = \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\, \sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

$$e^{tA} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} x_0 \cos(\omega t) + v_0 \frac{1}{\omega} \sin(\omega t) \\ -\omega x_0 \sin(\omega t) + v_0 \cos(\omega t) \end{pmatrix}$$

$$x(t) = x_0 \cos(\omega t) + v_0 \frac{1}{\omega} \sin(\omega t), \qquad v(t) = -\omega x_0 \sin(\omega t) + v_0 \cos(\omega t)$$

Ornstein-Uhlenbeck

$$dx = \lambda x\, dt + \sigma\, dW_t, \qquad x(0) = x_0$$

$$x(t) = x_0 + \lambda \int_0^t x(s)\, ds + \sigma\, W_t$$

$$\boxed{x(t) = e^{\lambda t} x_0 + \lambda\, \sigma \int_0^t e^{\lambda(t-s)}\, W_s\, ds + \sigma\, W_t}$$

$$x_0 + \lambda \int_0^t \Big(e^{\lambda s} x_0 + \lambda\, \sigma \int_0^s e^{\lambda(s-r)}\, W_r\, dr + \sigma\, W_s \Big)\, ds + \sigma\, W_t$$

$$= x_0 + \int_0^t \lambda e^{\lambda s} x_0\, ds + \lambda \int_0^t \Big(\lambda\, \sigma \int_0^s e^{\lambda(s-r)}\, W_r\, dr + \sigma\, W_s \Big)\, ds + \sigma\, W_t$$

$$= x_0 + e^{\lambda t} x_0 - x_0 + \lambda \int_0^t \Big(\lambda\, \sigma \int_0^s e^{\lambda(s-r)}\, W_r\, dr + \sigma\, W_s \Big)\, ds + \sigma\, W_t$$

$$= e^{\lambda t}x_0 + \lambda\,\sigma \int_0^t \Big(\lambda \int_0^s e^{\lambda(s-r)}\, W_r\, dr + W_s\Big)\, ds + \sigma\, W_t$$

$$= e^{\lambda t}x_0 + \lambda\,\sigma \int_0^t \Big(\lambda \int_0^s e^{\lambda(s-r)}\, W_r\, dr\Big)\, ds + \lambda\,\sigma \int_0^t W_s\, ds + \sigma\, W_t$$

$$= e^{\lambda t}x_0 + \lambda\,\sigma \int_0^t \Big(\lambda e^{\lambda s} \int_0^s e^{-\lambda r}\, W_r\, dr\Big)\, ds + \lambda\,\sigma \int_0^t W_s\, ds + \sigma\, W_t$$

$$= e^{\lambda t}x_0 + \lambda\,\sigma \Big[e^{\lambda s} \int_0^s e^{-\lambda r}\, W_r\, dr\Big|_0^t - \int_0^t \Big(e^{\lambda s}\, e^{-\lambda s}\, W_s\Big)\, ds\Big] + \lambda\,\sigma \int_0^t W_s\, ds + \sigma\, W_t$$

$$= e^{\lambda t}x_0 + \lambda\,\sigma \Big[\int_0^s e^{\lambda(s-r)}\, W_r\, dr\Big]_0^t + \sigma\, W_t = x(t)$$

$$\lambda \int_0^t e^{\lambda(t-s)}\, W_s\, ds + W_t = \int_0^t e^{\lambda(t-s)}\, dW_s \sim N\Big(0, \frac{e^{2\lambda t}-1}{2\lambda}\Big)$$

$$\boxed{x(t) = e^{\lambda t}x_0 + \sigma \int_0^t e^{\lambda(t-s)}\, dW_s}$$

$$x(t) = x_0 - \lambda \int_0^t x(s)\, ds + \sigma\, W_t$$

$$\boxed{x(t) = e^{-\lambda t}x_0 + \sigma \int_0^t e^{-\lambda(t-s)}\, dW_s}$$

$$\sigma \int_0^t e^{-\lambda(t-s)}\, dW_s \sim N\Big(0, \frac{\sigma^2}{2\lambda}(1-e^{2\lambda t})\Big)$$

$$x_0 \sim N\Big(0, \frac{\sigma^2}{2\lambda}\Big) \qquad \text{ed indipendente dal processo di Wiener}$$

$$x(t) \sim N\Big(0, \frac{\sigma^2}{2\lambda}\Big)$$

È un processo stazionario.

Vasicek

$$dx = \lambda(\mu - x)\, dt + \sigma\, dW_t, \qquad x(0) = x_0$$

$$x(t) = x_0 + \lambda \int_0^t (\mu - x(s))\, ds + \sigma\, W_t$$

$$z(t) = x(t) - \mu, \qquad z(0) = x_0 - \mu = z_0$$

$$z(t) = z_0 - \lambda \int_0^t z(s)\, ds + \sigma\, W_t$$

$$z(t) = e^{-\lambda t} z_0 + \sigma \int_0^t e^{-\lambda(t-s)}\, dW_s$$

$$x(t) = \mu + e^{-\lambda t}(x_0 - \mu) + \sigma \int_0^t e^{-\lambda(t-s)}\, dW_s$$

$$x(t) = \mu(1 - e^{-\lambda t}) + e^{-\lambda t} x_0 + \sigma \int_0^t e^{-\lambda(t-s)}\, dW_s$$

$\mathbb{E}(x(t)) = \mu(1 - e^{-\lambda t})$ se $\mathbb{E}(x_0) = 0$.

$$x_0 \sim N\Big(\mu, \frac{\sigma^2}{2\lambda}\Big) \qquad \text{ed indipendente dal processo di Wiener}$$

$$x(t) \sim N\Big(\mu, \frac{\sigma^2}{2\lambda}\Big)$$

È un processo stazionario.

Brownian bridge

$$dx = -\frac{x-b}{T-t}\, dt + dW_t, \quad x(0) = a$$

$$x(t) = a - \int_0^t \frac{x(s) - b}{T-s}\, ds + W_t, \qquad 0 \le t < T$$

$z(t) = x(t) - W_t$, $z(0) = a$

$$z(t) = a - \int_0^t \frac{x(s) - b}{T-s}\, ds, \qquad 0 \le t < T$$

$$z'(t) = -\frac{z(t) - b + W_t}{T-t}, \qquad 0 \le t < T$$

$$z'(t) + \frac{z(t)}{T-t} = \frac{b - W_t}{T-t}, \qquad 0 \le t < T$$

$$\frac{z'(t)}{T-t} + \frac{z(t)}{(T-t)^2} = \frac{b-W_t}{(T-t)^2}, \qquad 0 \le t < T$$

$$\Big(\frac{z(t)}{T-t}\Big)' = \frac{b-W_t}{(T-t)^2}, \qquad 0 \le t < T$$

$$\frac{z(t)}{T-t} - \frac{a}{T} = \int_0^t \frac{b-W_s}{(T-s)^2}\, ds, \qquad 0 \le t < T$$

$$x(t) = \frac{a}{T}(T-t) + (T-t)\int_0^t \frac{b-W_s}{(T-s)^2}\, ds + W_t, \qquad 0 \le t < T$$

$$x(t) = a\frac{T-t}{T} + b\Big(1 - \frac{T-t}{T}\Big) - (T-t)\int_0^t \frac{W_s}{(T-s)^2}\, ds + W_t, \qquad 0 \le t < T$$

$$W_t - (T-t)\int_0^t \frac{W_s}{(T-s)^2}\, ds = \int_0^t \frac{T-t}{T-s} dW_s$$

$$x(t) = a\Big(1 - \frac{t}{T}\Big) + b\,\frac{t}{T} + \int_0^t \frac{T-t}{T-s} dW_s \qquad 0 \le t < T$$

Per de l'Hôpital abbiamo che

$$(T-t)\int_0^t \frac{W_s}{(T-s)^2}\, ds = \frac{\displaystyle\int_0^t \frac{W_s}{(T-s)^2}\, ds}{\displaystyle\frac{1}{T-t}} \to \lim_{t\to T} \frac{\displaystyle\frac{W_t}{(T-t)^2}}{\displaystyle\frac{1}{(T-t)^2}} = W_T$$

da cui

$$\lim_{t\to T}\int_0^t \frac{T-t}{T-s} dW_s = 0$$

$a = b = 0$, $T = 1$

$$x(t) = \int_0^t \frac{1-t}{1-s} dW_s \qquad 0 \le t < 1$$

$z \sim N(0,1)$ ed indipendente da W:

$$\beta(t) := x(t) + t\, z = t\, z + \int_0^t \frac{1-t}{1-s} dW_s \qquad 0 \le t < 1$$

È un processo di Wiener tale che $z = \beta(1)$ e quindi $x(t) = \beta(t) - t\,\beta(1)$.