Filters

1 Linear Systems

In this chapter we outline the tenets of analysis of linear systems, which is among the most relevant applications of the Fourier and Laplace transforms. This will involve the point of view and the language of engineering of *signal processing,* of *communication theory,* and of *control theory* (the latter is also a mathematical topic). In addressing these issues the mathematician is called to reformulate several notions and results in precise and rigorous form.

First, we shall introduce some basic properties that systems may or may not fulfill: time-invariance, causality, stability, and so on. We shall characterize them in terms of response to the unit impulse (also called impulsive response of the system), and shall use integral transforms. We shall also characterize the action of time-invariant systems as the convolution between the input function and the response to the unit impulse. In the space of frequencies, [1] namely after Fourier or Laplace transform of the input-output relation, the impulsive response will correspond to the product of the spectra (namely, of the transformed functions).

We shall see that systems can be arranged in series, in parallel, and in retroaction (or feedback).

Signals and systems. The notion of *system* occurs in many disciplines: physics, chemistry, biology, engineering, economics, and so on. We can define a system as a unitary whole, that can exchange information with the rest of the world. The system will be regarded as a *black box;* by this we mean that the internal structure of the system itself will not be considered.

We can define a *signal* as a function of time that represents a physical quantity, and evolves in time. In this context a *system* is represented by an operator that transforms an input signal to an output signal. [2] Here we shall just deal with complex-valued signals.

Let us consider a *one-dimensional system,* that maps a time-dependent a complex input signal to a time-dependent complex output signal. [3] One distinguishes between discrete and analogical systems, in which time is discrete or continuous, respectively, and is represented by $n \in \mathbb{Z}$ or $t \in \mathbb{R}$. One also distinguishes between deterministic and stochastic systems: depending of whether the response is uniquely determined by the input. In this presentation we shall just deal with deterministic analogical systems.

Input and output signals will be constrained to appropriate linear spaces of complex-valued functions of time, with a space Φ of *admissible inputs* and a space Ψ of outputs. Each system will be associated with an operator $L : \Phi \to \Psi : f \mapsto g$, which will be often identified with the system itself. The spaces Φ and Ψ normally include $L^p(\mathbb{R})$, with $p \in [1, \infty]$ (most frequenly $p = 1$ or 2 or ∞).

Linearity and continuity. Linear systems correspond linear operators $L : \Phi \to \Psi$ between

[1] As we did above, by frequency we refer to the *angular* frequency.

[2] For instance, a control system transforms the *control* (the input) to an *observable* (the output).

[3] This also called a SISO *(single input and single output)* system. We shall not deal with MIMO *(multiple input and multiple output)* systems (in this second case the scalar transfer function [defined ahead] is replaced by a matrix-valued transfer function).

(complex) linear spaces. [4] Thus

$$L\bigg(\sum_{i=1}^{N} a_i f_i\bigg) = \sum_{i=1}^{N} L(a_i f_i) \qquad \forall f_1, ..., f_N \in \Phi, \forall a_1 ..., a_N \in \mathbb{C}, \forall N \in \mathbb{N}. \tag{1.1}$$

The extension to infinite linear combinations and to integrals requires a limit operation, and for this purpose the spaces Φ and Ψ must also be equipped with a topology. More precisely, they will be *topological linear spaces,* [5] and the linear operator L will also be continuous.

Examples. Linear systems that are studied in physics and engineering include the following one:
(i) *Amplifiers,* that are represented by a system of the form $L(f) = Af$, with amplification factor $A > 0$. These systems are called amplifiers in strict sense if $A > 1$, attenuators if $A < 1$.
(ii) *Derivators,* that are characterized by $L(f) = f'$.
(iii) *Integrators,* that are characterized by $L(f)' = f$,
(iv) So-called *RLC circuits.* These are electrical systems, that are constructed by arranging either in series or in parallel (see ahead) elementary constituents: resistors, inductors, condensers [6] and an electrical generator, which typically generates an alternating-current. In case of serial interconnection of those elements, the system is characterized by an ODE of the form

$$Ly''(t) + Ry'(t) + \frac{y(t)}{C} = f(t). \tag{1.2}$$

Here by $y = y(t)$ we denote the electrical charge, by $f = f(t)$ an applied tension, by L the inductance, by R the resistance, by C the capacitance. (**??**) is called the *state equation* of the system, [7]
(v) *Delayed circuits,* that are characterized by a state equation of the form

$$L(f) = f(\cdot - \tau) =: f_\tau \qquad \text{for some fixed } \tau > 0. \tag{1.3}$$

(vi) So-called *rheological systems.* These are mechanical systems that consist in a mass, a spring and a dashpot (i.e., a viscous element). The corresponding state equation is a second order ODE, like (**??**). The analogy between RLC circuits and rheological systems is much exploited in engineering.

Here we shall not consider nonlinear systems, which include elements with state equation of the form $[L(f)](t) = \varphi(f(t))$, for some nonlinear mapping $\varphi : \mathbb{C} \to \mathbb{C}$.

Next we outline the main properties that linear systems may or may not fulfill: time-invariance, causality, stability, reality.

[4] We use the notation L because of linearity. This L should not be confused either with that of L^p-spaces, or with that of Laplace transform (that we denote by $\mathcal{L}$).

[5] This means that they are linear spaces as well as topological spaces, and that the linear operations (namely sum and multiplication by a scalar) are continuous with respect to that topology. Here we shall not address the theory of these spaces.

[6] These are often referred to as *passive electric elements,* since they generate no energy. Resistors dissipate (electric) energy, whereas inductors and condensers first store and then deliver it.

[7] Differential systems are often defined by more general ODEs of the form $P(t, D)y = Q(t, D)f$, with $P(t, D)$ and $Q(t, D)$ linear differential operators with coefficients that may depend on t. The order of the operator $P(t, D)$ is called the order of the system.

Time-invariance. First, let us define the time-shift operator $\Phi \to \Phi : f \mapsto f_\tau$ as in (**??**). A system L is called *time-invariant* (or *stationary)* whenever

$$f_\tau \in \Phi, \qquad L(f_\tau) = L(f)_\tau \qquad \forall f \in \Phi, \forall \tau \in \mathbb{R}. \tag{1.4}$$

This means that a time-shift of the input induces the same shift of the response. This occurs for instance for systems that are represented by *autonomous* differential equations (i.e., differential equations with constant coefficients). The above examples of linear systems are time-invariant.

Causality. A system L is called *causal* whenever

$$f_1(t) = f_2(t) \quad \forall t < \tau \quad \Rightarrow \quad (Lf_1)(t) = (Lf_2)(t) \quad \forall t < \tau \qquad \forall \tau \in \mathbb{R}, \forall f_1, f_2 \in \Phi. \tag{1.5}$$

Because of the linearity of L, this holds iff

$$f(t) = 0 \quad \forall t < \tau \quad \Rightarrow \quad (Lf)(t) = 0 \quad \forall t < \tau \qquad \forall \tau \in \mathbb{R}, \forall f \in \Phi. \tag{1.6}$$

On the other hand, a signal $f(t)$ is called causal if $f(t) = 0$ for any $t < 0$. Causal systems thus map causal signals to causal signals. Ahead we shall see why the term causality is used for systems and signals, referring to apparently different properties.

All physically realizable systems are causal, so that some authors use the term *realizable* as a synonymous of causal.

Stability. Let $p \in [1, +\infty]$. A system $L : \Phi \to \Phi$ is called *stable in L^p* (or *p-stable)* [8] if it maps $\Phi \cap L^p$ to L^p, and if there exists a constant $C_p > 0$ such that

$$\|Lf\|_{L^p} \leq C_p \|f\|_{L^p} \qquad \forall f \in \Phi \cap L^p.$$

In particular, for $p = \infty$ one speaks of BIBO *(bounded input bounded output)* stability.

Amplifiers, delayed circuits and RLC circuits are stable in L^p, at variance with derivators and integrators.

Further properties. Time-invariance, causality and stability are the main properties that a linear system may or may not satisfy. However there are also other properties.

A system L is called *real* if it transforms real signals to real signals, that is, for any $f \in \Phi$

$$f(t) \in \mathbb{R} \quad \forall t \quad \Rightarrow \quad (Lf)(t) \in \mathbb{R} \qquad \forall t \in \mathbb{R}. \tag{1.7}$$

For instance, amplifiers, derivators, integrators, and delayed circuits are all real. RLC circuits and rheological systems need not be so, since the solution of a second-degree algebraic equation may be complex (this depends on the parameters). This is one of the reasons why the theory mainly deals with complex-valued systems.

For instance, a system is called *instantaneous* (or *memoryless)* if at each instant $L(f)$ only depends on the value of the input f at the same instant. In this case the system L is of the form $[L(f)](t) = \varphi(f(t))$ for some function $\varphi : \mathbb{C} \to \mathbb{C}$. Amplifiers are the only memoryless linear systems.

A system is called *invertible* if the operator L is invertible.

[8] In mathematics the term *stability* has several meanings, and in the present case analysts rather say that L is a bounded operator in L^p.

One also distinguishes between analog and digital systems, between discrete-time (or sampled) and continuous-time filters, between deterministic and stochastic filters, and so on. In some cases the operator L is defined for functions defined for a.e. $t \in \mathbb{R}$ (more shortly, "$q\forall t \in \mathbb{R}$"), and accordingly the state equation holds a.e..

Impulsive response. We shall be especially concerned with linear systems that act on a topological linear space Φ, that includes Lebesgue integrable signals and also impulsive signals (e.g., linear combinations of translated Dirac masses). Compactly supported distributions then look as a natural environment. More precisely, we shall deal with linear continuous systems $L : \Phi \to \mathcal{D}'$ assuming that $\mathcal{E}' \subset \Phi$. When using integral transforms, we shall also assume that $\Phi \subset \mathcal{S}'$ and $L : \Phi \to \mathcal{S}'$. [9]

Proposition 1.1 *Let $\mathcal{E}' \subset \Phi$ and $L : \Phi \to \mathcal{D}'$ be a linear and continuous system. Let us define the response to the shifted impulse $\delta_\tau = \delta_0(\cdot - \tau)$:*

$$\bar{h}(t, \tau) := [L(\delta_\tau)](t) \qquad q\forall t, \tau \in \mathbb{R}. \tag{1.8}$$

If

$$\bar{h}(t, \cdot) \in L^1 \qquad q\forall t \in \mathbb{R}, \tag{1.9}$$

then [10]

$$[L(f)](t) = \langle \bar{h}(t, \tau), f(\tau) \rangle \qquad q\forall t \in \mathbb{R}, \forall f \in \Phi \cap L^\infty. \tag{1.10}$$

The signal $\bar{h}$ thus completely characterizes the system L.

Proof. (Omitted) □

Remark 1.2 The integral (**??**) is a convolution if $\bar{h}(t, \tau)$ only depends on $t - \tau$. More precisely, if

$$\exists h \in L^1 : \qquad \bar{h}(t, \tau) = h(t - \tau) \quad \forall t, \tau \in \mathbb{R}, \tag{1.11}$$

then

$$[L(f)](t) = \int_{\mathbb{R}} f(\tau) h(t - \tau)\, d\tau = (f * h)(t) \qquad q\forall t \in \mathbb{R}, \forall f \in \Phi \cap L^\infty. \qquad \square \tag{1.12}$$

For instance, for any $\tau \in \mathbb{R}$ the response to the shifted unit step $H_\tau := H(\cdot - \tau)$ is

$$[L(H_\tau)](t) = \int_{\mathbb{R}} H_\tau(r) \bar{h}(t, r)\, dr = \int_\tau^{+\infty} \bar{h}(t, r)\, dr \qquad q\forall t \in \mathbb{R}. \tag{1.13}$$

The next two theorems characterize the main properties of linear systems in terms of the response to the shifted unit impulse.

[9] This selection of the functional set-up is a choice of this presentation. Most of the engineering literature ([Hsu], [Papoulis], [Roubine], and so on) does not specify the domain of definition of the linear system.

[10] In order to assist the Reader, we display the *integration* variable, although in general $\bar{h}$ is a distribution.

Theorem 1.3 *Let $\mathcal{E}' \subset \Phi$, $L : \Phi \to \mathcal{D}'$ be linear and continuous, and assume that the response $\bar{h}$ to the shifted impulse fulfills* (**??**). *Then:*

(i) L is time-invariant iff

$$\bar{h}(t,\tau) = \bar{h}(t-\tau, 0) \qquad q\forall t,\tau \in \mathbb{R}. \tag{1.14}$$

In this case the system L is characterized by the function

$$h(t) := \bar{h}(t,0) \; \big(= [L(\delta_0)](t)\big) \qquad q\forall t \in \mathbb{R}. \tag{1.15}$$

(ii) L is causal iff $\bar{h}(t,\tau) = 0$ for any $t < \tau$. In this case

$$[L(f)](t) = \int_0^t f(\tau)\bar{h}(t,\tau)\, d\tau \qquad \forall t \in \mathbb{R}, \forall f \in \Phi. \tag{1.16}$$

* **Proof.** For each issue just the "if" part needs a proof.

(i) For any $f \in \Phi$, by using (**??**) and changing the integration variable, we have ARed: usare funzioni regolari dense.

$$\begin{aligned} [L(f_\sigma)](t) &= \int_{\mathbb{R}} f_\sigma(\tau)\bar{h}(t,\tau)\, d\tau \overset{(??)}{=} \int_{\mathbb{R}} f_\sigma(\tau) h(t-\tau)\, d\tau \\ &= \int_{\mathbb{R}} f(\tau-\sigma) h(t-\tau)\, d\tau = \int_{\mathbb{R}} f(s) h(t-\sigma-s)\, ds \\ &= [L(f)](t-\sigma) = [L(f)]_\sigma(t) \qquad q\forall t \in \mathbb{R}, \forall \sigma \in \mathbb{R}. \end{aligned} \tag{1.17}$$

(ii) Let us assume that $f \in \Phi$ with $\text{supp}(f) \subset \mathbb{R}^+$, and fix any $\widetilde{\tau} \in \mathbb{R}$. If $\bar{h}(t,\tau) = 0$ for all $t < \widetilde{\tau}$, then the integrand of (**??**) does not vanish only if $\widetilde{\tau} \le \tau \le t$; The integral then vanishes for all $t < \widetilde{\tau}$. □

Remarks 1.4 (i) Because of (**??**), at any instant t the output thus does not depend on the input at successive instants. This is what we usually mean by causality.

Because of parts (i) and (ii) of Theorem **??**, a time-invariant linear system is causal iff its impulsive response $L(\delta_0)$ vanishes for any negative time. This explains why a signal is called causal iff it vanishes for any negative time.

(ii) A linear system L is real iff $\bar{h}(t,\tau) \in \mathbb{R}$ for a.e. $t, \tau \in \mathbb{R}$. □

Theorem 1.5 *Let $L^p \cap \mathcal{E}' \subset \Phi$, let $L : \Phi \to \mathcal{D}'$ be a a time-invariant linear and continuous system, and assume that the response $\bar{h}$ to the shifted impulse fulfills* (**??**). *If $L(\delta_0) \in L^1$ then L is stable in L^p for any $p \in [1, +\infty]$.*

Proof. This follows from the Young Theorem **??** on convolution: let us recall the synthetical formula $L^p * L^1 \subset L^1$. □

Remark 1.6 The instability on a linear system does not exclude the possibility of applying the integral transforms. For instance, in order to apply the Fourier transform to $f \in \Phi$ and to Lf, it suffices that $\Phi \subset \mathcal{S}'$ and that $L : \mathcal{S}' \to \mathcal{S}'$ is continuous. □

Theorem 1.7 *Let $\mathcal{E}' \subset \Phi$, and $L : \Phi \to \mathcal{D}'$ be a stable, linear and continuous system. Then L is time-invariant iff*

$$L(f) = L(\delta_0) * f \qquad \forall f \in \Phi. \tag{1.18}$$

Because of this representation, L is then called a *convolution system.*

Proof. If L is time-invariant, then by part (i) of Theorem **??** (**??**) reads

$$L : f \mapsto \int_{\mathbb{R}} f(\tau)h(t-\tau)\, d\tau = (f * h)(t) = (f * L(\delta_0))(t). \tag{1.19}$$

Conversely it is clear that $f \mapsto L(\delta_0) * f$ is time-invariant. □

2 Filters and Transfer Functions

Above we dealt with linear systems that act on time-dependent signals. In this section we define filters and their transfer functions; we then use the Fourier and Laplace transforms to represent the filters themselves as frequency-dependent signals, and reformulate some of their properties in this new set-up. We thus derive some relations among filters, impulsive signals, Fourier and Laplace transforms, transfer functions, and so on.

Filters. The term *filter* may have two meanings. In *signal processing* (which is an engineering discipline) a filter is a physical device that modifies the signal, typically by acting on its spectral components. This process is called *filtering.* In *signal analysis* (which is essentially a mathematical discipline)

$$\text{a filter is a time-invariant linear and continuous system.} \tag{2.1}$$

In these notes we shall just deal with this second notion, and assume that the unit impulse δ_0 is an admissible input. Examples include amplifiers, RLC circuits, derivators, integrators, delayed circuits, and the many systems that are constructed by composing these basic elements according to certain rules, as we shall see ahead.

As we saw, a filter L is stable if $h := L(\delta_0) \in L^1$ (some authors state the converse property, too); L is causal (real, resp.) iff h is causal (real, resp.). Causal filters are effectively realizable. These include *differential filters* $f \mapsto u$, that are characterized by a state equation of the form

$$P(D)u = Q(D)f \qquad P(D), Q(D) \text{ being polynomial differential operators} \tag{2.2}$$

(thus with constant coefficients). The corresponding impulsive response $h = L(\delta_0)$ fulfills the differential equation $P(D)h = Q(D)\delta_0$. These filters include for instance RLC circuits, which are differential filters of order two.

Most typical filters select certain frequencies of the input. For instance, low-pass, high-pass, band-pass filters select a range of frequencies, that are respectively either below or above a certain threshold, or confined to a prescribed band.

Fourier transform for stable filters. First, we apply the Fourier transform to stable filters with (possibly noncausal) integrable signals as input, and briefly illustrate energetic aspects. Afterwards we use the Laplace transform to deal with causal signals, and with possibly unstable filters.

Let $L : \Phi \to \mathcal{S}'$ be a filter such that $h := L(\delta_0) \in L^1$. As we saw, the system is then stable and is characterized by the impulsive response h. As $\mathcal{F} : L^1 \to C^0_b$, we can then define the *impulsive response in frequency:* [11]

$$\mathcal{H}(\omega) := [\mathcal{F}(h)](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(t)e^{-i\omega t}\, dt \qquad \forall \omega \in \mathbb{R}. \tag{2.3}$$

[11] The notation $\mathcal{H}$ should not be confused with that of the Heaviside function H.
Here we denote the dual variable by ω instead of ξ, as it is often done in the engineering literature.

This function is also called the *transfer function*, or the *system function*, or the *spectrum* of the system. (The term *transfer function* is most often used for the Laplace transform of h, when admissible.)

As we saw, the Fourier transform of a signal is called its *spectrum.* It is natural to use the same term for the system and for the signal: indeed, as we shall see ahead (see Theorem XXX), the role of the system is somehow comparable to that of the input signal.

Next we introduce some standard terms taken from the engineering literature. Denoting by $R(\omega)$ and $X(\omega)$ respectively the real and imaginary part of $\mathcal{H}(\omega)$, by $A(\omega)$ its modulus, and by $\varphi(\omega)$ its phase, we have

$$\mathcal{H}(\omega) = R(\omega) + iX(\omega) = A(\omega)e^{i\varphi(\omega)} \qquad \forall \omega \in \mathbb{R}. \tag{2.4}$$

The functions A and φ are respectively called the *amplitude spectrum* and the *phase spectrum* of the system. If L is real then the functions R and A are even, whereas X and φ are odd. [12]

By the results of Chapter [Fourier], $h = \mathcal{F}^{-1}(\mathcal{H})$ in the sense of tempered distributions. If $\mathcal{H} \in \mathcal{F}(L^1)$, then we can express this function as an integral

$$h(t) = [\mathcal{F}^{-1}(\mathcal{H})](t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{H}(\omega)e^{i\omega t}\, d\omega \qquad q\forall t \in \mathbb{R}.$$

If $\mathcal{H} \in \mathcal{F}(L^2)$ this must be understood as a principal value.

Theorem 2.1 *(Fundamental theorem of filters – I) Let $\mathcal{E}' \subset \Phi \to \mathcal{S}'$, let $L : \Phi \to \mathcal{S}'$ be a filter, and assume that $h := L(\delta_0) \in L^1$. Let $p \in [1,2]$ and $f \in \Phi \cap L^p$ (assumed nonempty), set $g = L(f)$, and define the Fourier transformed functions $F := \mathcal{F}(f)$, $G := \mathcal{F}(g)$, $\mathcal{H} := \mathcal{F}(h)$. Then* [13]

$$G(\omega) = \sqrt{2\pi}\, \mathcal{H}(\omega)\, F(\omega) \qquad \forall \omega \in \mathbb{R}. \tag{2.5}$$

Proof. By Proposition 3.1 [Fourier section] and Theorem **??**, $\mathcal{H} \in C_b^0$ and $g \in L^p$. By the Hausdorff-Young Theorem then $F, G \in L^{p'}$ with $p' = p/(p-1)$ if $p \neq 1$ and $\infty' = 1$.

(**??**) follows from the extension of the Convolution Theorem to L^p spaces. □

The formula (**??**) obviously yields

$$|G(\omega)|^2 = 2\pi\, |\mathcal{H}(\omega)|^2\, |F(\omega)|^2 \qquad \forall \omega \in \mathbb{R}. \tag{2.6}$$

Remarks 2.2 (i) The symmetry with which signals and filter occur in (**??**) and (**??**) allows one to deal with filters on the same footing ad signals. [14]

(ii) For any function $\mathcal{H}$ in the range of the Fourier transform $\mathcal{F} : L^1 \to C_b^0$, one can conceive (and design!) a filter that has $\mathcal{H}$ as its system function. ARed: is this a mathematical issue?

Energy of signals. Let us consider an electrical current flowing along a conducting wire having unit resistance. If $f(t)$ is the current density at any instant $t \in \mathbb{R}$ and $f \in L^2$, then $\|f\|^2_{L^2}$ is interpreted as the total energy of the signal. Accordingly, $|f(t)|^2$ is regarded as the density of

[12] In engineering $A(\omega)$ and $\phi(\omega)$ are also called the *gain* and the *phase shift* of the system at the frequency ω.

[13] By Theorem **??**, $g \in L^p$. Hence $F, G, \mathcal{H} \in C_b^0$.
In literature this formula often occurs without the factor $\sqrt{2\pi}$, because of the different scaling in the definition of the Fourier transform.

[14] This is especially significative, and explains why the term *causal* is used for signals that vanish for any negative time. ARed: explain

energy at the instant t. For any real a, b with $a < b$, $\int_a^b |f(t)|^2\,dt = \int \chi_{]a,b[}|f(t)|^2\,dt$ is then the energy of the signal in the time interval $]a, b[$. (Signals of L^2 are then named *energy signals.)*

As $|f(t)|^2$ can be interpreted as the time density of the energy of a signal, the Plancherel Theorem

$$\int_{\mathbb{R}} |F(\omega)|^2\,d\omega = \int_{\mathbb{R}} |f(t)|^2\,dt \qquad (F := \mathcal{F}(f)) \tag{2.7}$$

suggests that the energy can similarly be partitioned with respect to frequency, with frequency density $|F(\omega)|^2$. This is interpreted as the density of energy of the signal at the frequency ω. The function $\omega \mapsto |F(\omega)|^2$ is then called the *energy spectrum* of the signal.

A system that fulfills the assumptions of Theorem **??** transforms the energy spectrum of a signal as in (**??**). The continuous and bounded function $2\pi\,|\mathcal{H}(\omega)|^2$ is then called the *power spectrum* of the system.

Note that in the relation (**??**) signal and system are set on the same footing.

Laplace transform of causal filters. So far in this section we assumed that $h := L(\delta_0) \in L^1$, and applied the Fourier transform. Next we rather assume that h is a causal Laplace-transformable distribution: $h \in \bar{D}_{\mathcal{L}}$. In this case L is a (possibly unstable) filter.

Let us denote the Laplace transform of h by $\mathcal{K}$:

$$\mathcal{K}(s) := [\mathcal{L}(h)](s) = \int_{\mathbb{R}} h(t)e^{-st}\,dt \qquad \forall s \in \mathbb{C}_{\lambda(h)}. \tag{2.8}$$

(This actually is a duality pairing: in section [Laplace transform of distributions] we saw the exact meaning of this definition.) We know that this function is analytic in the half-plane of convergence $\mathbb{C}_{\lambda(h)}$. As we did for the Fourier transform, here also we call $\mathcal{K}$ *transfer function,* or *system function,* or also *spectrum* of the system.

The fundamental theorem of filters here reads as follows.

Theorem 2.3 *(Fundamental theorem for filters – II) Let $\Phi \subset \mathcal{S}'$, and let $L : \Phi \to \mathcal{S}'$ be a (possibly unstable) filter such that $h := L(\delta_0) \in \bar{D}_{\mathcal{L}}$. Let $f \in \Phi \cap D_{\mathcal{L}}$, set $g := L(f)$, and define the Laplace transformed functions $F := \mathcal{L}(f)$, $G := \mathcal{L}(g)$, $\mathcal{K} := \mathcal{L}(h)$ (all analytic functions). Then*

$$G(s) = \mathcal{K}(s)\,F(s) \qquad q\forall s \in \mathbb{C}_{\lambda(g)},\ \textit{with}\ \lambda(g) \le \max\{\lambda(h).\lambda(f)\}. \tag{2.9}$$

This obviously entails

$$|G(s)|^2 = |\mathcal{K}(s)|^2\,|F(s)|^2 \qquad q\forall s \in \mathbb{C}_{\lambda(g)}, \tag{2.10}$$

so that here the energy spectrum of the system is $|\mathcal{K}(s)|^2$.

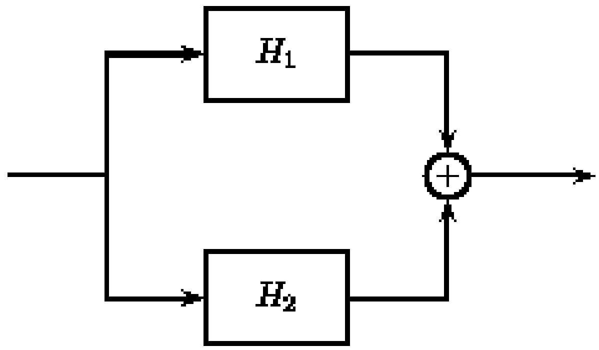
3 Filter Composition

In this section we arrange filters in series, in parallel, and in retroaction (or feedback): for each of these composed system we derive the state equation and the transfer function.

Let two filters $L_1, L_2 : \Phi \to \mathcal{S}'$ be defined on the same topological linear subspace Φ of $\mathcal{S}'$, and be characterized by the respective impulsive responses $h_1, h_2 \in L^1$. First, we deal with integrable inputs. So let $f \in \Phi \cap L^1$ and set

$$F := \mathcal{F}(f), \quad g_i := L_i(f), \quad G_i := \mathcal{F}(g_i), \quad \mathcal{H}_i := \mathcal{F}(h_i) \quad (i = 1, 2). \tag{3.1}$$

The filters L_1 and L_2 can be composed according to the three fundamental arrangements, that we next outline: in parallel, in series and in feedback.



(i) Parallel arrangement. The picture above represents the composed system L. This is defined by the map

$$L : f \mapsto g = L_1(f) + L_2(f) \qquad \forall f \in \Phi,$$

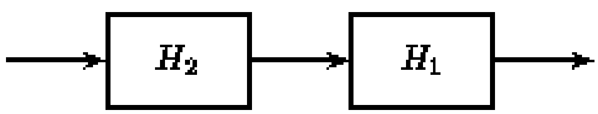
and is characterized as follows in terms of the impulsive responses of L_1 and L_2:

$$g(t) = [(h_1 + h_2)*f](t) \qquad \forall t \in \mathbb{R}, \tag{3.2}$$

$$G(\omega) = \sqrt{2\pi}\,[\mathcal{H}_1(\omega) + \mathcal{H}_2(\omega)]F(\omega) \qquad \forall \omega \in \mathbb{R}. \tag{3.3}$$

L is thus a filter with impulsive response

$$h = h_1 + h_2 \quad \text{in time,} \qquad \mathcal{H} = \sqrt{2\pi}\,(\mathcal{H}_1 + \mathcal{H}_2) \quad \text{in frequency.}$$



(ii) Series arrangement. The picture above represents the composed system L. This is defined by the map

$$L : f \mapsto g = L_2(L_1(f)) \qquad \forall f \in \Phi,$$

and is characterized as follows in terms of the impulsive responses of L_1 and L_2: [15]

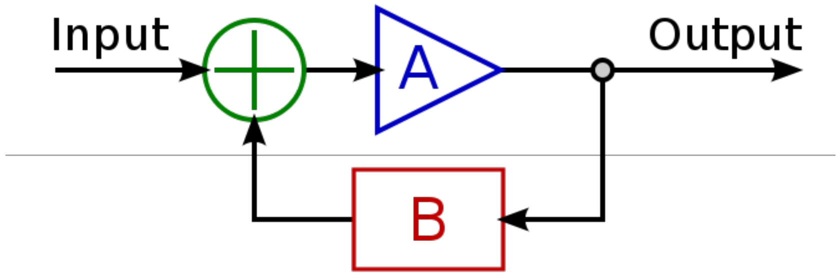
$$g(t) = [h_2 * h_1 * f](t) \qquad \forall t \in \mathbb{R}, \tag{3.4}$$

$$G(\omega) = 2\pi\,\mathcal{H}_2(\omega)\,\mathcal{H}_1(\omega)\,F(\omega) \qquad \forall \omega \in \mathbb{R}. \tag{3.5}$$

L is thus a filter with impulsive response

$$h = h_2 *\ h_1 \quad \text{in time,} \qquad \mathcal{H} = 2\pi\,\mathcal{H}_2\,\mathcal{H}_1 \quad \text{in frequency.}$$

[15] We write $h_1 * h_2 * f$ since the convolution is associative.



(iii) Feedback arrangement. [16] The picture above represents the composed system L (herewith A and B in place of L_1 and L_2). Two filters L_1 and L_2 are connected in negative feedback if the output of L_1 enters L_2, and the difference between the input and this second output enters L_1. [17] This corresponds to the state equation

$$L : f \mapsto g(t) = L_1(f + L_2(g))(t) \qquad \forall t \in \mathbb{R}, \tag{3.6}$$

that is, in terms of the impulsive responses of L_1 and L_2:

$$g(t) = [h_1 * (f + h_2 * g)](t) \qquad \forall t \in \mathbb{R}. \tag{3.7}$$

By applying the Fourier transform, we get

$$G(\omega) = \sqrt{2\pi}\,[\mathcal{H}_1(F + \sqrt{2\pi}\mathcal{H}_2 G)](\omega) \qquad \forall \omega \in \mathbb{R}, \tag{3.8}$$

that is,

$$G(\omega) = \frac{\sqrt{2\pi}\,\mathcal{H}_1(\omega)}{1 - 2\pi\,\mathcal{H}_1(\omega)\,\mathcal{H}_2(\omega)} F(\omega) \qquad \forall \omega \in \mathbb{R}, \tag{3.9}$$

provided that the denominator does not vanish for any ω. In this case the impulsive response in frequency of the composed filter L thus reads

$$\mathcal{H}(\omega) = \frac{\sqrt{2\pi}\,\mathcal{H}_1(\omega)}{1 - 2\pi\,\mathcal{H}_1(\omega)\,\mathcal{H}_2(\omega)} \qquad \forall \omega \in \mathbb{R},$$

and by inverting the Fourier transform we get the impulsive response in time:

$$h(t) = \mathcal{F}^{-1}\left(\frac{\sqrt{2\pi}\,\mathcal{H}_1(\omega)}{1 - 2\pi\,\mathcal{H}_1(\omega)\,\mathcal{H}_2(\omega)}\right)(t) \qquad \forall t \in \mathbb{R}. \tag{3.10}$$

Here sufficient conditions for the stability of the feedback system L are less obvious than for the two previous cases.

There are two types of feedback:

(i) *Negative feedback.* In this case the signal fed back from the output reduces the effect of the input signal. This stabilizes the system.

(ii) *Positive feedback.* In this case the signal fed back from the output intensifies the effect of the input signal. This destabilizes the system.

[16] This arrangement is at the basis of many servo-mecchanisms and control systems.

There are two common classes of control action: open loop and closed loop. In an open-loop control system, the control action from the controller is independent of the process variable. An example of this is a central heating boiler controlled only by a timer. In a closed-loop control system, the control action from the controller depends on the actual value of the process variable. In the case of the boiler analogy, a thermostat would monitor the temperature, and feed back a signal so that the controller output maintains the building temperature close to that set on the thermostat.

[17] Here any delays is neglected.