Minimization

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1 Minimization

A Topological Result. Let S be a topological space and $J : S \to]-\infty, +\infty]$; here we deal with the problem of finding $x_0 \in S$ such that $J(x_0) = \inf J$. By the classical Weierstrass theorem, if S is a compact topological space and J is continuous then this problem has a solution. However these assumptions also yield the existence of a maximum, and this suggests that they might be redundant. Indeed the following results tells us that, when dealing with minimization, continuity can be replaced by lower semicontinuity.

Let us denote the set of minimizers of J by $\mathcal{M}_J$; that is,

$$\mathcal{M}_J := \{x \in S : J(x) = \inf J\}.$$

Theorem 1.1 *(Topological Result) Let S be a nonempty topological space and $J : S \to]-\infty, +\infty]$.*

(i) If J is lower semicontinuous and for some $\tilde{a} \in \mathbf{R}$ the sublevel set $S_{\tilde{a}} := \{x \in S : J(x) \leq \tilde{a}\}$ is nonempty and compact, then $\mathcal{M}_J$ is nonempty and compact.

(ii) If J is sequentially lower semicontinuous and $S_{\tilde{a}}$ is nonempty and sequentially compact for some $\tilde{a} \in \mathbf{R}$, then $\mathcal{M}_J$ is nonempty and sequentially compact. (1)

Proof. At first let us assume (i). The family $\mathcal{F} := \{S_a : a \leq \tilde{a}, S_a \neq \emptyset\}$ consists of closed subsets of $S_{\tilde{a}}$, by the lower semicontinuity of J, and the intersection of any finite subfamily is nonempty. Hence, by the compactness of $S_{\tilde{a}}$, the intersection of the whole family, $\cap\mathcal{F}$, is nonempty and compact. It is easy to see that $\cap\mathcal{F} = \mathcal{M}_J$.

Let us now assume (ii). By the definition of $\inf J$, there exists a sequence such that $J(x_n) \to \inf J$; possibly dropping a finite number of terms, we have $\{x_n\} \subset S_{\tilde{a}}$. By the sequential compactness of the latter set, there exist $x \in S$ and a subsequence $\{x_{n'}\}$ such that $x_{n'} \to x$. As J is sequentially lower semicontinuous, this entails that $J(x) = \inf J$. The sequential compactness of $\mathcal{M}_J$ can similarly be checked. (1) □

Some Results in Banach Spaces. In order to apply the previous general result, one must choose an appropriate topology; for instance, in Banach spaces the weak and the strong topology are at disposal. It then appears that the lower semicontinuity of J and the compactness of $S_{\tilde{a}}$ are competing requirements: the first one forces the topology in S to be sufficiently strong, the second one induces it to be sufficiently weak. A compromise between these opposite exhigencies must then be reached.

We recall the reader three classical compactness properties.

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(1) In passing, we illustrate a topological construction. In any topological space (X, τ) the sequential topology, denoted by seq-τ, that is associated to τ is defined as follows. We say that a set $B \subset X$ is closed w.r.t. seq-τ if and only if it contains the limit of any sequence of elements of B that converges w.r.t. τ. This sequential topology is finer than the original topology, i.e. $\tau \subset$ seq-τ. If the topology τ is metrizable then it coincides with seq-τ. This sequential topology allows to restate several sequential properties of the original topology. E.g., any set is sequentially compact w.r.t. τ if and only if it is compact w.r.t. seq-τ; a function is sequentially lower semicontinuous w.r.t. τ if and only if it is lower semicontinuous w.r.t. seq-τ; and so on.

(1) This type of argument based on the use of minimizing sequences is often referred to as the *direct method of the calculus of variations*. Dealing with a differentiable functional J defined on a topological vector space, the *indirect method* consists in studying the minimization problem via the *Euler equation* $J'(u) = 0$.

The direct method might also be used for part (i), just replacing sequences by nets and subsequence by cofinal subnets.

Lemma 1.2

(i) (Weak Star Compactness) In the dual of a Banach space, any bounded set is weakly star relatively compact (that is, its closure in the weak star topology is weakly star compact). [In a reflexive Banach space, any bounded set is then weakly relatively compact.]

(ii) (Weak Star Sequential Compactness) In the dual of a separable Banach space, any bounded set is weakly star relatively sequentially compact (that is, any bounded sequence has a weakly star convergent subsequence).

(iii) (Weak Sequential Compactness) In a reflexive Banach space, any bounded sequence is weakly compact.

This lemma and Theorem 1.1 yield the following result.

Corollary 1.3 *(Minimization via Weak Compactness) Let B be a Banach space and $J : B \to]-\infty, +\infty]$ be such that*

$$\exists \tilde{a} \in \mathbf{R} : \{x \in B : J(x) \leq \tilde{a}\} \ \ \textit{is nonempty and bounded.} \tag{1.1}$$

(i) If B is the dual of a Banach space and J is weakly star lower semicontinuous, then $\mathcal{M}_J$ is nonempty and weakly star compact.

(ii) If B is the dual of a separable Banach space and J is sequentially weakly star lower semicontinuous, then $\mathcal{M}_J$ is nonempty and sequentially weakly star compact.

(iii) If B is a reflexive Banach space and J is sequentially weakly lower semicontinuous, then $\mathcal{M}_J$ is nonempty and sequentially weakly compact.

(1.1) holds whenever J is proper and the following stronger condition, called *coerciveness,* is fulfilled:

$$J(v_n) \to +\infty \quad \text{as} \quad \|v_n\|_B \to +\infty. \tag{1.2}$$

(This is tantamount to the boundedness of all sublevel sets.)

As (strong) semicontinuity is less restrictive than weak semicontinuity, the next result is often applied.

• **Corollary 1.4** *(Minimization via Convexity) Let B be a reflexive Banach space, $J : B \to]-\infty, +\infty]$ be convex and lower semicontinuous, and (1.1) be fulfilled.*

Then $\mathcal{M}_J$ is nonempty, weakly compact and convex (hence closed). If J is strictly convex then there exists only one minimizer.

Proof. By Mazur's Theorem (a consequence of the Hahn-Banach theorem), J is weakly lower semicontinuous. The properties of $\mathcal{M}_J$ then follow from Corollary 1.3(iii).

If there exist two distinct points $u_1, u_2 \in \mathcal{M}_J$, then $J((u_1 + u_2)/2) < [J(u_1) + J(u_2)]/2 = \inf J$ by the strict convexity of J. This contradicts the definition of $\mathcal{M}_J$. □

This result fails if B is nonreflexive, even if it is the dual of a separable Banach space. In fact in this case convex lower semicontinuous functionals need not be weakly star lower semicontinuous.

The above results are easily extended to minimization over a set K, just by replacing J by $J + I_K$, where I_K is the indicator function of K.

Well-Posedness. Many problems can be regarded as a transformation, $\mathcal{T}$, from a space of data, $\mathcal{D}$, to a space of solutions, $\mathcal{S}$. In several cases the solution exists and is unique, i.e., $\mathcal{T}$ is a single-valued mapping $\mathcal{D} \to \mathcal{S}$. If $\mathcal{D}$ and $\mathcal{S}$ are topological spaces, it is also of interest to see whether $\mathcal{T}$ is continuous; in this case the problem is said to be *well-posed* (in the sense of Hadamard).

Dealing with minimization problems, another concept of well-posedness is also useful. Let S be a separated topological space. The problem of minimizing a proper functional $J : S \to]-\infty, +\infty]$

is said *well-posed in the sense of Tychonov* iff any minimizing sequence converges to a minimum point:

$$\forall \{u_n\} \subset S, \ \text{ if } \ J(u_n) \to \inf J \ \text{ then } \ \exists u \in S : u_n \to u, \ J(u) = \inf J. \tag{1.3}$$

This holds iff a minimum point exists and any minimizing sequence is convergent, and entails the uniqueness of the minimum point.

For instance, let us consider the following functionals $J_i : \mathbf{R} \to]-\infty, +\infty]$

$$J_1(u) := \begin{cases} u^2(u-1)^2 & \text{if } u \neq 1, \\ 1 & \text{if } u = 1, \end{cases} \qquad J_2(u) := u^2 e^{-u}, \qquad J_3(u) := u^2(u-1)^2.$$

None of them is well-posed in the sense of Tychonov. J_1 and J_2 may be regarded as rather pathological. Both functionals have one and only one minimum point, $u = 0$. However, $\{u_n := 1 + 1/n\}$ is a minimizing sequence for J_1, but it converges to a point which is not of minimum; on the other hand, $\{u_n := n\}$ is a minimizing sequence for J_2, but it does not converge.

The functional J_3 has two minimum points, at variance with J_1 and J_2. The minimization of J_3 is not well-posed in the sense of Tychonov, but this functional does not look as especially pathological. A weaker concept of well-posedness has indeed been proposed. A minimization problem is said *well-posed in the generalized sense of Tychonov* iff a converging subsequence can be extracted from any minimizing sequence, and this subsequence tends to a minimum point:

$$\begin{aligned} &\forall \{u_n\} \subset S, \ \text{ if } \ J(u_n) \to \inf J, \ \text{ then} \\ &\exists u \in S, \exists \{u_{n'}\} \subset \{u_n\} : u_{n'} \to u, \ J(u) = \inf J. \end{aligned} \tag{1.4}$$

J_3 is well-posed in this generalized sense, whereas J_1 and J_2 are not.

Exercise. Let B be a Banach space, $J : B \to]-\infty, +\infty]$, and $I := \{x \in B : J(x) = \inf J\}$. Show that if J is convex (lower semicontinuous, resp.) then I is convex (closed, resp.).

2. The theorem of Lions-Stampacchia for variational inequalities.

Let H be a Hilbert space, with scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Let $f \in H$, set $J(v) := \frac{1}{2}\|v\|^2 - (f, v)$ for any $v \in H$, and consider the problem of minimizing J in a nonempty closed convex set $K \subset H$. Note that J is Fréchet-differentiable in H, with differential $J'(v) = v - f$ for any $v \in H$.

Exercise. Show that $u \in H$ minimizes J in K iff

$$(u - f, u - v) + I_K(u) - I_K(v) \leq 0 \qquad \forall v \in H, \tag{2.1}$$

or equivalently

$$u \in K \quad \text{ and } \quad (u - f, u - v) \leq 0 \quad \forall v \in K. \tag{2.2}$$ □

As $J(v) := \frac{1}{2}\|v - f\|^2 - \frac{1}{2}\|f\|^2$, it is easily seen that (2.2) holds iff u minimizes the distance of f from K, that is, iff u is the projection of f onto K, $u = P_K(f)$. We claim that the operator $P_K : H \to K$ is *nonexpansive,* i.e.,

$$\|P_K(f_1) - P_K(f_2)\| \leq \|f_1 - f_2\| \qquad \forall f_1, f_2 \in H. \tag{2.3}$$

This can easily be checked by a standard procedure: let us write (2.2) for f_1 and $u_1 := P_K(f_1)$ (f_2 and $u_2 := P_K(f_2)$, resp.), take $v = u_2$ ($v = u_1$, resp.), and then sum the two inequalities. This yields $\|u_1 - u_2\|^2 \leq (f_1 - f_2, u_1 - u_2) \leq \|f_1 - f_2\| \, \|u_1 - u_2\|$, whence (2.3) follows.

More generally, one can consider a linear bounded operator $A : H \to H$, and minimize the functional $\hat{J} : v \mapsto \frac{1}{2}(Av, v) - (f, v)$ in K, for any fixed $f \in H$. This problem is equivalent to the variational inequality

$$u \in K \quad \text{and} \quad (A_s u - f, u - v) \leq 0 \;\; \forall v \in K; \tag{2.4}$$

here $A_s := (A + A^*)/2$ is the *symmetric part* of A. [(2)] More generally, we consider a nonlinear operator $\mathcal{A} : H \to H$ and the variational inequality

$$u \in K \quad \text{and} \quad (\mathcal{A}(u) - f, u - v) \leq 0 \;\; \forall v \in K. \tag{2.5}$$

We shall assume that $\mathcal{A}$ is Lipschitz continuous and *strongly monotone* in K, that is,

$$\exists L > 0 : \forall u, v \in K, \; \|\mathcal{A}(u) - \mathcal{A}(v)\| \leq L\|u - v\|, \tag{2.6}$$

$$\exists \alpha > 0 : \forall u, v \in K, \; (\mathcal{A}(u) - \mathcal{A}(v), u - v) \geq \alpha\|u - v\|^2. \tag{2.7}$$

In general (2.5) is not equivalent to the minimization of any functional. This fails even in the linear case whenever the operator is not symmetric.

Theorem 2.1 *(Lions-Stampacchia) Let K be a nonempty closed convex subset of a real Hilbert space H, and $\mathcal{A} : H \to H$ fulfill (2.6) and (2.7). Then for any $f \in H$ there exists one and only one solution of (2.5), and this depends Lipschitz continuously on f.*

Proof. For any $\rho > 0$, the inequality of (2.5) is equivalent to

$$\big(u - [u - \rho(\mathcal{A}(u) - f)], u - v\big) \leq 0 \qquad \forall v \in K,$$

which also reads $u = P_K(u - \rho(\mathcal{A}(u) - f))$. We claim that for a suitable $\rho > 0$ the mapping $T = T_\rho : v \mapsto v - \rho(\mathcal{A}(v) - f)$ is a contraction, i.e., there exists $a < 1$ such that $\|T(u) - T(v)\| \leq a\|u - v\|$ for any $u, v \in H$. Denoting by L the Lipschitz constant of $\mathcal{A}$, for any $v_1, v_2 \in H$, actually we have

$$\begin{aligned}
\|T(v_1) - T(v_2)\|^2 &= \|v_1 - v_2\|^2 + \rho^2\|\mathcal{A}(v_1) - \mathcal{A}(v_2)\|^2 - 2\rho(\mathcal{A}(v_1) - \mathcal{A}(v_2), v_1 - v_2) \\
&\leq \|v_1 - v_2\|^2 + \rho^2 L^2\|v_1 - v_2\|^2 - 2\rho\alpha\|v_1 - v_2\|^2 \\
&= (1 + \rho^2 L^2 - 2\rho\alpha)\|v_1 - v_2\|^2.
\end{aligned}$$

Hence T is a contraction if $0 < \rho < 2\alpha/L^2$. Therefore $P_K \circ T$ is also a contraction, and by the fixed-point Banach theorem $P_K \circ T$ has one and only one fixed point. Therefore the variational inequality (2.5) has one and only one solution.

The Lipschitz continuity of the *solution operator* $f \mapsto u$ is a straightforward consequence of (2.7). □

The latter theorem entails the following classical result.

Corollary 2.2 *(Lax-Milgram) Let H be a real Hilbert space, and $\mathcal{A} : H \to H$ be linear, bounded and strongly monotone. Then for any $f \in H$ there exists one and only one $u \in H$ such that $\mathcal{A}u = f$, and the mapping $f \mapsto u$ is linear and continuous.*

If the operator $\mathcal{A}$ is also symmetric, then the thesis easily follows from the classical Riesz-Fréchet theorem for the representation of the dual of a Hilbert space. [Ex]

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(2) Any linear and bounded operator A acting in a Hilbert space can be written as the sum of its symmetric and anti-symmetric parts: $A = A_s + A_a$, where $A_s := (A + A^*)/2$ and $A_a := (A - A^*)/2$. Here A^* is the adjoint of A, which operates in $H^* = H$. Note that $(Av, v) = (A_s v, v)$ for any $v \in H$.

3. Γ-Convergence

Here we confine ourselves to metric spaces, although Γ-convergence may also be defined in the more general framework of topological spaces.

Definitions. Let (X, d) be a metric space, a sequence $\{f_n\}$ and f be functions $X \to [-\infty, +\infty]$, and $u \in X$. We say that f_n *Γ-converges to* f (in (X, d)) at u, and write $f(u) = \Gamma \lim_{n\to\infty} f_n(u)$ (or $f_n(u) \xrightarrow{\Gamma} f(u)$), iff

$$(i) \text{ for any sequence } \{u_n\} \text{ in } X, \text{ if } u_n \to u \text{ then } \liminf_{n\to\infty} f_n(u_n) \geq f(u), \tag{3.1}$$

$$(ii) \begin{array}{l} \text{there exists a sequence } \{u_n\} \text{ in } X \\ \text{such that } u_n \to u \text{ and } \limsup_{n\to\infty} f_n(u_n) \leq f(u) \\ \text{(equivalently, } f_n(u_n) \to f(u), \text{ by part (i)).} \end{array} \tag{3.2}$$

We say that f_n Γ-converges to f whenever this holds for any $u \in X$.

If in (3.2) we replace $\limsup_{n\to\infty} f_n(u_n)$ by $\liminf_{n\to\infty} f_n(u_n)$, then (i) and (ii) define the *inferior Γ-limit*

$$f(u) = \min\left\{ \liminf_{n\to\infty} f_n(u_n) : u_n \to u \text{ in } X \right\} =: \Gamma\!-\!\liminf_{n\to\infty} f_n(u) \qquad \forall u \in X.$$

On the other hand, if in (3.1) we replace $\liminf_{n\to\infty} f_n(u_n)$ by $\limsup_{n\to\infty} f_n(u_n)$, then (i) and (ii) define the *superior Γ-limit*

$$f(u) = \min\left\{ \limsup_{n\to\infty} f_n(u_n) : u_n \to u \text{ in } X \right\} =: \Gamma\!-\!\limsup_{n\to\infty} f_n(u) \qquad \forall u \in X.$$

The two latter limits exist for any sequence $\{f_n\}$, and of course

$$\Gamma \liminf_{n\to\infty} f_n(u) \leq \Gamma \limsup_{n\to\infty} f_n(u) \qquad \forall u \in X.$$

Moreover, f_n Γ-converges iff these limits are equal, and in this case their common value coincides with the Γ-limit. There is no symmetry between the inferior and superior Γ-limits; actually, in general

$$\Gamma \liminf_{n\to\infty} (-f_n) \neq -\Gamma \limsup_{n\to\infty} f_n, \qquad \text{whence} \quad \Gamma \lim_{n\to\infty} (-f_n) \neq -\Gamma \lim_{n\to\infty} f_n,$$

whenever the Γ-limit exists. The functions $\Gamma \liminf_{n\to\infty} f_n$ and $\Gamma \limsup_{n\to\infty} f_n$ are indeed both lower semicontinuous. [] The same then applies to the Γ-limit, whenever it exists. (3)

Proposition 3.1 *Let $\{f_n\}$ be a sequence of functions $X \to [-\infty, +\infty]$. Then:*

(i) If $f_n \xrightarrow{\Gamma} f$, then f is lower semicontinuous.

(ii) If $\{f_n\}$ is a nondecreasing (nonincreasing, resp.) sequence of functions $X \to [-\infty, +\infty]$, then

$$f_n \xrightarrow{\Gamma} \overline{\sup\{f_n\}} \qquad (f_n \xrightarrow{\Gamma} \overline{\inf\{f_n\}}, \textit{ resp.}). \ [Ex] \tag{3.3}$$

Hence

$$\text{if} \quad f_n = f_0 \ \forall n, \quad \text{then} \quad f_n \xrightarrow{\Gamma} \bar{f}_0. \tag{3.4}$$

Here are some further properties of Γ-convergence.

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(3) For any function $f : X \to [-\infty, +\infty]$ we shall denote its lower semicontinuous regularized function by $\bar{f}$.

Proposition 3.2 *(Comparison with the Pointwise Limit) Let $\{f_n\}$ be a sequence of functions $X \to [-\infty, +\infty]$, $f : X \to [-\infty, +\infty]$, and $f_n \xrightarrow{\Gamma} f$. Then:*

$$\Gamma\!-\!\liminf_{n\to\infty} f_n(u) \le \liminf_{n\to\infty} f_n(u) \quad \forall u \in X, \tag{3.5}$$

$$\Gamma\!-\!\limsup_{n\to\infty} f_n(u) \le \limsup_{n\to\infty} f_n(u) \quad \forall u \in X. \tag{3.6}$$

Equalities hold whenever (denoting by $\mathcal{U}(u)$ the family of neighbourhoods of u)

$$\forall \varepsilon > 0, \exists U \in \mathcal{U}(u) : \forall v \in U, f_n(v) \ge f_n(u) - \varepsilon \qquad \textit{(equi-lower-semicontinuity)}. \tag{3.7}$$

Proposition 3.3 *(Compactness) Let (X, d) be a separable metric space (i.e., which admits a countable base of open sets), and $\{f_n\}$ be a sequence of functions $X \to [-\infty, +\infty]$. Then there exists a Γ-convergent subsequence of $\{f_n\}$. []*

The next result entails a useful characterization of Γ-convergence as a variational convergence: whenever $f_n \xrightarrow{\Gamma} f$, the limit of any (converging) sequence of minimizers of f_n minimizes f.

Proposition 3.4 *(Minimization) Let (X, d) be a metric space, and $\{f_n\}$ is a sequence of functions $X \to]-\infty, +\infty]$ such that $f_n \xrightarrow{\Gamma} f$. Assume that* $\inf f_n > -\infty$ *for any n, and that $\{u_n\} \subset X$ and $u \in X$ are such that*

$$f_n(u_n) \le \inf f_n + \frac{1}{n} \quad \forall n, \qquad u_n \to u \quad \textit{in} \quad X. \tag{3.8}$$

Then

$$\inf f_n \to \inf f, \qquad \liminf_{n\to\infty} f_n(u_n) \ge f(u) = \inf f. \tag{3.9}$$

Proof. By Γ-convergence, $f(u) \le \liminf_{n\to\infty} f_n(u_n)$; moreover, for any $v \in X$, there exists $\{v_n\} \subset X$ such that $v_n \to v$ in X and $f_n(v_n) \to f(v)$. By (3.8), $f_n(u_n) \le f(v_n) + \frac{1}{n}$, for any n. Therefore

$$f(u) \le \liminf_{n\to\infty} f_n(u_n) \le \lim_{n\to\infty} f_n(v_n) = f(v) \qquad \forall v \in X. \qquad \square$$

Examples. Here X coincides with $\mathbf{R}$, equipped with the Euclidean metric.

(i) Let us set $f_n(x) := (-1)^n x$ for any $x \in \mathbf{R}$ and any n. Then

$$\Gamma \liminf_{n\to\infty} f_n(x) = -|x|, \qquad \Gamma \limsup_{n\to\infty} f_n(x) = |x| \qquad \forall x \in \mathbf{R}.$$

(ii) Let us set $f_n(x) := \cos(nx)$ for any $x \in [-\pi, \pi]$ and any n. Then $f_n \xrightarrow{\Gamma} -1$, although $f_n \to 0$ weakly in $L^1(-\pi, \pi)$, and the pointwise limit exists only for $x = 0$. Note that

$$(-1 =)\ \Gamma \lim_{n\to\infty} f_n = \Gamma \lim_{n\to\infty} (-f_n) \ne -\Gamma \lim_{n\to\infty} f_n\ (= 1).$$

(iii) Let us set $f_n(x) := x\cos(nx)$ and $f(x) := -|x|$ for any $x \in [-\pi, \pi]$ and any n. Then $f_n \xrightarrow{\Gamma} f$. Notice that f is even, although any f_n is odd.

(iv) Let set $f_n(x) := nx\exp(nx)$ for any $x \in \mathbf{R}$ and any n. Then $f_n \xrightarrow{\Gamma} f$, where

$$f(x) := \begin{cases} 0 & \text{if } x < 0, \\ -1/e & \text{if } x = 0, \quad \text{[Ex]} \\ +\infty & \text{if } x > 0, \end{cases}$$

although $f_n(0) \to 0$.

(v) Let $\{q_n\}$ be an enumeration of $\mathbf{Q}$, and set

$$\begin{cases} f_n(x) := 0 & \text{if } x = q_m \text{ for some } m \geq n \\ f_n(x) := 1 & \text{otherwise} \end{cases} \qquad \forall x \in \mathbf{R}.$$

Then $f_n(x) \nearrow 1$ for a.a. $x \in \mathbf{R}$, but $f_n \xrightarrow{\Gamma} 0$, by part (ii) of Proposition 3.1.

An Example Related to Perimeters.

Theorem 3.5 *(Modica and Mortola) Let us set $\mathcal{S} := \{v \in L^1(\mathbf{R}^N)\colon v \in \mathbf{Z}$ a.e. in $\mathbf{R}^N\}$ ($n \in \mathbf{N}$), and for any $v \in L^1(\mathbf{R}^N)$*

$$f_n(v) := \begin{cases} \displaystyle\int_{\mathbf{R}^N} \Big(\frac{1}{n}|\nabla v|^2 + n\sin^2(\pi v)\Big)dx & \textit{if } v \in H^1(\mathbf{R}^N), \\ +\infty & \textit{otherwise,} \end{cases} \tag{3.10}$$

$$f(v) := \begin{cases} \displaystyle\frac{4}{\pi}\int_{\mathbf{R}^N} |\nabla v| & \textit{if } v \in \mathcal{S}, \\ +\infty & \textit{otherwise.} \end{cases} \tag{3.11}$$

Then $f_n \xrightarrow{\Gamma} f$ in $L^1(\mathbf{R}^N)$. []

Remarks about the Proof. Let us set $\varphi(v) := 2\int_0^v \sin(\pi\xi)d\xi$ for any $v \in \mathbf{R}$; thus $\varphi(v) = 4v/\pi$ for any $v \in \mathbf{Z}$. Notice that

$$\frac{1}{n}|\nabla v|^2 + n\sin^2(\pi v) = \Big(\frac{1}{\sqrt{n}}|\nabla v| - \sqrt{n}\sin(\pi v)\Big)^2 + 2|\nabla v||\sin(\pi v)| \geq |\nabla\varphi(v)| \quad \text{a.e. in } \mathbf{R}^N, \forall v \in H^1(\mathbf{R}^N). \tag{3.12}$$

If $\{u_n\}$ is a sequence in $H^1(\mathbf{R}^N)$, $u \in \mathcal{S}$, and $u_n \to u$ in $L^1(\mathbf{R}^N)$, then we have

$$\liminf_{n\to\infty} f_n(u_n) \geq \liminf_{n\to\infty} \int_{\mathbf{R}^N} |\nabla\varphi(u_n)|dx \geq \int_{\mathbf{R}^N} |\nabla\varphi(u)| = f(u). \tag{3.13}$$

It is easy to see that $f_n(u_n) \to +\infty$ whenever $u_n \to u \notin \mathcal{S}$ $L^1(\mathbf{R}^N)$. Part (i) of the definition of Γ-convergence is thus fulfilled. The most delicate step of the argument is to construct a sequence $\{u_n\}$ as required in (3.2). □

Corollary 3.6 *Let $\{f_n\}$ and f be defined as in the latter theorem, and $\{u_n\}$ be such that $f_n(u_n) = \inf f_n$ for any n. Then there exists $u \in \mathcal{S}$ such that, possibly extracting a subsequence,*

$$u_n \to u \qquad \textit{strongly in } L^1(\mathbf{R}^N). \tag{3.14}$$

By Proposition 3.2 this entails that $f_n(u_n) \to f(u) = \inf f$.

Proof. By (3.12), $\int_{\mathbf{R}^N} |\nabla\varphi(u_n)| \leq f_n(u_n)$ for any n; by (3.2), $\{f_n(u_n)\}$ is bounded from above. By the compactness of the injection $BV(\mathbf{R}^N) \subset L^1(\mathbf{R}^N)$, (3.14) then follows. □

The Γ-convergence can be extended to topological spaces.

Convergence in the Sense of Kuratowski.

Definition. Let (X, d) be a metric space, $\{A_n\}$ be a sequence of subsets of X, and $A \subset X$. We say that A_n *converges to A in the sense of Kuratowski* iff:
(i) for any $u \in X$, for any sequence $\{u_n \in A_n\}$, if $u_n \to u$, then $u \in A$;
(ii) for any $u \in A$, there exists a sequence $\{u_n \in A_n\}$ such that $u_n \to u$.

On the basis of the following result, Γ-convergence has also been named *epi-convergence.*

Proposition 3.7 *Let (X, d) be a metric space, and $\{f_n\}$ be a sequence of functions $X \to [-\infty, +\infty]$. Then $f_n \xrightarrow{\Gamma} f$ iff $\operatorname{epi}(f_n) \to \operatorname{epi}(f)$ in the sense of Kuratowski in $X \times \mathbf{R}$. Moreover, for any sequence of subsets $\{A_n\}$ of X, $I_{A_n} \xrightarrow{\Gamma} I_A$ iff $A_n \to A$ in the sense of Kuratowski in X. []*