Continuous Propositional Modal Logic

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Abstract

We introduce a propositional many-valued modal logic which is an extension of the Continuous Propositional Logic to a modal system. Otherwise said, we extend the minimal modal logic **K** to a Continuous Logic system. After introducing semantics, axioms and deduction rules, we establish some preliminary results. Then we prove the equivalence between consistency and satisfiability. As straightforward consequences, we get compactness and an approximated completeness theorem, in the vein of Continuous Logic.

1. Introduction

In this paper we introduce a modal extension of Continuous Propositional Logic (see [4] or [3]). Indeed, the system that we present can equally be regarded as a continuous extension (in the sense of Continuous Logic) of Propositional Modal Logic **K** (see [5]). Throughout this paper, $[0,1]$ denotes the real unit interval.

Modal logic has a long tradition in mathematical logic. Continuous Logic is quite new in its current formulation. For this reason we will spend a few words on it. Continuous Logic was preceded by Łukasiewicz's $[0,1]$–valued logic, Chang and Keisler's Continuous Model Theory and the Henson's logic of positive bounded formulas. It aims at providing a suitable logic to deal with first order versions of higher order structures that come equipped with a metric space structure: the so called *metric structures* that cover virtually all structures that are of interest to functional analysis, but also probability structures. Each continuous function from $[0,1]^n$ to $[0,1]$, where n is any natural number, plays the role of an n-ary connective and the classical quantifiers are replaced by the quantifiers sup and inf. Continuous Predicate Logic has a nice model theory. See [2].

In this paper we are not concerned with the model theoretic aspects of Continuous Logic. We rather want to regard it as a logic *per se* and to introduce a modal extension of its propositional fragment. We recall that the latter is an extension of $[0,1]$–valued Łukasiewicz logic. Hence we will be dealing with an infinitely valued logic.

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Ours is a first step towards a logic merging predicate continuous logic and a suitable modal (or temporal) system. There are some motivations for doing it. Firstly a possible intrinsic interest of such a logic. Secondly, we would like to devise a modal system that can formally express "metric" statements. To explain what we mean, let us consider the simplest example of a metric structure: a complete bounded metric space $\langle M, d \rangle$. Deformations of the set M (topological, physical, etc. . .) might be described by a family of metrics indexed on some set endowed with a binary relation. Let us assume that this set represents time and that the relation is a partial ordering, not necessarily linear, describing the time flow. Thus we have a system which nondeterministically evolves along time. Let $m, m' \in M$. We may want to be able to express properties like "from now on, no matter what deformation will occur, the distance between m and m' will always be less or equal to 1/2"; "at some future time instant m and m' will collapse on each other" (i.e. their distance will reduce to zero) or, when A and B are definable subregions of M, "from some future time instant onwards, the distance between A and B will always be greater than 1/3" (notice that the latter involves a quantification in the sense of continuous logic). The more the underlying metric structure is rich (a Banach space, a Banach algebra, a C^*-algebra. . .), the more the expressive power increases and the examples become interesting. Mostly, but not exclusively, from the mathematical viewpoint. In the future, we aim at a reasonably well-behaved predicate logic which can formally express statements like those above.

Going back to the propositional logic of this paper, the main obstacle in proving completeness is to make a modal system and a continuous one work well together. Indeed they have been conceived for different purposes. Techniques for establishing completeness which are available for each of the systems may not be sufficiently compatible. Actually, in Section 4 we prove an *approximated* completeness result (see Corollary 21 below). As pointed out in the comment at the beginning of Section 4, in presence of the Continuos Logic component, this is the best that we can achieve.

Comparison with the existing literature. Many valued and fuzzy systems with modal operators have been and are extensively studied. In the following we comment on some contributions that are related to our work. We believe that a thorough comparison with the extensive literature on many valued modal logics and fuzzy modal systems is beyond the scope of this paper.

Among the early investigations, we mention [9, 10]. Fitting's motivation was to devise a system for dealing with opinions of experts with a dominance relation among them. In [10] he proves a completeness theorem for a modal system where the formulas, as well as the accessibility relations, take values in the same finite Heyting algebra. To this end, Fitting introduces an adaptation of the canonical model construction in modal logic which is essentially the one used in this paper.

It is easily verified that the standard ordering naturally induces a Heyting algebra structure on the $[0,1]$ interval, but this is not the algebraic structure that we consider in this paper. As we already mentioned, the propositional logic underlying Continuous Logic is the $[0,1]$–valued Łukasiewicz logic. For this reason, we are concerned with the standard MV–algebra structure on $[0,1]$, rather than with its natural Heyting algebra structure. Loosely speaking, we may say that we aim at merging modal and continuous systems, rather than modal and intuitionistic ones.

At this point we stress that the semantics of the operator $\Box$ in a many valued modal logic is the natural extension of the Kripke semantics to a many-valued setting. It is the one adopted in all the literature. Different semantics can be assigned to $\Box$ in fuzzy systems, depending on the intended meaning of $\Box$.

We have already mentioned that, in [10], the accessibility relations are many–valued. In this paper they will always be $\{0,1\}$–valued (briefly: crisp). This is needed to have a *normal* modal component in our system. In this regard, Remark 4 in Section 3 below shows that, if we allow for many–valued accessibility relations, the validity of the characteristic axiom (also known as the *normal* axiom) of modal logic **K** fails with respect to the proposed semantics. The normality of the modal component plays a crucial role in our main result.

The non-validity of the normal axiom when the accessibility relations are many-valued is pointed out also in [6]. In the latter, the authors investigate minimum many-valued modal logics for the modal operator $\Box$ defined on top of logics of finite residuated lattices. They succeed in finding such logics in a number of cases depending on the class of Kripke frames (the accessibility relations may take values on the whole residuate lattice under consideration or on its idempotent elements, etc. . .) and on the kind of logical consequence relation (local or global). We anticipate that, in this paper, we will prove a local theorem. In our opinion the local consequence relation (see (3) after Remark 1 below) is better suited to describe the evolution of a metric structure.

Same as in [10], in [6] canonical constants (i.e names for elements of the underlying lattice) occur in the language, for technical reasons. Finiteness of the lattice plays a crucial role. A canonical model construction is used.

The part of [6] which is most related to our work is Section 5. Local and global complete axiomatizations are provided therein for a modal extension of the logic of an arbitrary *finite* MV-chain, under the assumption of crisp accessibility relations. Actually the authors point out that the same result had been previously obtained in [8] with respect to the global consequence relation.

More interesting to us, in [8, Section 6] Hansoul and Teheux prove a completeness result with respect to the local logical consequence relation for classes of many-valued modal logics extending modal logic **K**. For a comparison with our system, let us fix among the logics studied in [8] the one based on the $[0,1]$-valued Łukasiewicz logic and let us denote it by **L**. The **L**-axiom system includes, in addition to an axiomatization of Łukasiewicz logic and the normal axiom of **K**, a number of axiom schemas whose intuitive meaning is that the necessity operator behaves well with respect to the MV-algebra structure. There are similar, but simpler, axiom schemas in our system (see A8 and A9 below). In order to get their completeness result, Hansoul and Teheux claim that they introduce a new infinitary deduction rule. In our opinion it would be more appropriate to say that they define a notion of "infinitary provability", denoted by $\vdash^{\infty}_{\mathbf{L}}$, but they do not extend the deductive system to fully accommodate the infinitary rule. Indeed the relation $\vdash^{\infty}_{\mathbf{L}}$ is defined in terms of infinitely many instances of the finitary provability relation $\vdash_{\mathbf{L}}$ (see [8, Definition 6.5]). Otherwise said, application of the infinitary rule is allowed at most once in a proof, at the very end of it.

The interested reader may find in [1] an extension of propositional continuos logic by means of an infinitary rule which allows to strengthen the approximate completeness theorem of continuous logic to an "exact" theorem. Unfortunately the resulting system lacks good proof-theoretic properties like, for instance, a deduction theorem extending that of the finitary system. In order to restore the validity of the latter theorem, ad hoc restrictions on the application of the infinitary rule need to be imposed.

Returning to a comparison with [8], apart from the different approach (the use of MV-algebra homomorphisms instead of maximal consistent sets), Hansoul and Teheux's canonical model construction is based on the same idea as in [9] and in our paper (see the definition of the canonical structure after Theorem 17 below). We stress that the canonical model construction is completely driven by the semantics of the modal operator $\square$. For this reason it is the natural adaptation of the canonical model construction relative to **K**. Furthermore, once the necessary translations from one formalization to the other have been made, it can be seen that the premisses of the infinitary deduction rule in [8] describe the same situation as the right-hand side of our approximated completeness theorem (Corollary 21). Roughly speaking, in both [8, Proposition 6.6] and Corollary 21, an equivalence between logical consequence and provability of a formula "up to any prescribed degree of accuracy" is established. In this author's opinion the meaning of such equivalence is better understood if stated in the continuos logic setting.

Apart from their intrinsic interest, Hansoul and Teheux seem not provide extra-logical motivations for developing their systems the way they do. They claim that they are guided by the will to consider many-valued modal systems for which algebraic tools already exist. In this regard, it is quite remarkable that different starting points lead to essentially the same result.

Eventually we mention [7]. In that paper the authors endow the members of a suitable class of logics, each based on the monoidal t-norm logic MTL, with a modal operator. They call such an operator a *truth stresser modality*. Indeed they are mostly concerned with fuzzy operators. The algebraic counterpart of each logic is given by some class of residuated lattices, where the modality $\square$ is interpreted by a unary operator. Hypersequent calculi are provided and completeness results with respect to algebras based on t-norms are established, together with related results. The class of logics under consideration is quite rich but does not include the $[0, 1]$-valued Łukasiewicz logic (see [7, Section 2]). For this reason, we are not able to make a thorough comparison with our work.

Structure of the paper. In Sections 2 and 3 we introduce the semantics and the deductive system respectively. As already said, structures are $[0, 1]$-valued Kripke structures and the semantics is the natural extension of that of propositional Continuous Logic to a modal framework. The axioms are those of Continuous Propositional Logic, together with the characteristic axiom of modal logic **K** and some other quite natural axioms. We will comment on those axioms just after introducing them. We have two inference rules: modus ponens and necessitation.

In Section 4 we prove the equivalence of consistency and satisfiability; the above mentioned approximated completeness theorem and compactness of our logic, as consequences of our main technical result (Theorem 18).

Eventually we discuss whether our system is a conservative extension of Łukasiewicz and of Continuous Logic and we point out a curious connection between modal logic **S5** and the use of $[0,1]$–valued accessibility relations.

2. Language, structures, semantics

Let $\Sigma_0 = \{P_i : i \in I\}$ be a set of pairwise distinct proposition symbols. Let $\Sigma_1 = \{\dot{-}, \neg, \frac{1}{2}\} \cup \{\Box\}$, where $\neg$ and $\frac{1}{2}$ are unary connectives, $\dot{-}$ is a binary connective and $\Box$ is a modal operator. As usual, we have the brackets are auxiliary symbols.

The set F of formulas is the least set that contains the proposition letters and is closed under application of the connectives and of $\Box$. We shall use abbreviations like $\varphi_1 \dot{-} \varphi_2 \dot{-} \ldots \dot{-} \varphi_n$, assuming that the missing brackets are associated to the left. We also assume that $\Box$ binds more tightly than all the connectives and that $\neg$ binds more tightly than $\dot{-}$.

We call *Łukasiewicz formula* (briefly: Ł–formula) a formula whose connectives are in the set $\{\dot{-}, \neg\}$. We call *Continuous formula* (briefly: CL–formula) a $\Box$–free formula. We denote by F_L and F_C the sets of Ł– and of CL–formulas respectively. We may use $\Diamond$ as an abbreviation for $\neg \Box \neg$.

Next we introduce an extension of Kripke structures. In the following, a *structure* is a triple

$$\langle M, r : M \times M \to \{0,1\}, v : M \times \Sigma_0 \to [0,1] \rangle$$

where M is a nomempty set, r and v are binary functions and $[0,1]$ is the real unit interval. The function r is the characteristic function of the accessibility relation between elements of M. We make the convention that, for all $k, m \in M$, the world k is accessible from m if and only if $r(m,k) = 0$. This convention agrees with the semantics described below, where 0 is the highest truth value.

When no confusion arises, we shall write just M to denote the above triple. If M is a singleton, we say that M is a *Łukasiewicz structure* (briefly: Ł–structure), or a *Continuous Logic structure* (briefly: CL–structure), depending on the setting.

Mapping $v : M \times \Sigma_0 \to [0,1]$ uniquely extends to a mapping, also denoted by the same name, $v : M \times F \to [0,1]$ as follows:

(1) $v(m, \frac{1}{2}\varphi) = \frac{1}{2}v(m,\varphi)$;
(2) $v(m, \neg\varphi) = 1 - v(m,\varphi)$;

(3) $v(m, \varphi \dot{-} \psi) = \max(v(m,\varphi) - v(m,\psi), 0))$;
(4) $v(m, \Box\varphi) = \sup\{v(k,\varphi) \dot{-} r(m,k) : k \in M\}$.

Next we comment on the semantics, for reader's convenience. As anticipated, in the current semantics the values 0 and 1 play the role of "absolute truth" and "absolute falsehood" respectively. The reals in the open interval $]0,1[$ play the role of intermediate truth values (the higher the real, the lower its degree of truth). Of course, we may reverse the role of 0 and 1 and modify the semantics accordingly. We stick to [3] so that we can directly refer to the results therein. Actually, the choice made in [3] (which was inherited from [2]) is related to a feature of continuous logic that we would like to describe. In its predicate version, it is a logic without equality. The natural many-valued counterpart of the equality predicate, relative to the domain A of a continuous logic structure, is a metric $d : A \times A \to [0,1]$. (Boundedness of the metric is a technical requirement that can be bypassed. See [2].) Equality between elements of A can be recovered from the metric by noticing that, for all $a, b \in A$,

$$a = b \iff d(a,b) = 0.$$

Hence the value 0 yields the highest degree of equality.

As noticed in [3, Remark 2.3], the connective $\dot{-}$ plays the role of reverse implication, in the sense that $\varphi \dot{-} \psi$ may be interpreted as "φ is implied by ψ". Actually, if the truth degree of φ is greater or equal to that of ψ with respect to some truth valuation v (remember that this means $v(\varphi) \leq v(\psi)$), then the truth degree of $\varphi \dot{-} \psi$ is the highest possible, i.e. $v(\varphi \dot{-} \psi) = 0$. In our opinion the choice of $\dot{-}$ as a primitive connective, rather than the implication, does lead to some notational and computational simplification. However, our main motivation for sticking to $\dot{-}$ is compliance with [3].

Concerning the role of the connective $\frac{1}{2}$, roughly speaking we want a set of connectives with the property that every real in the unit interval is the limit of a converging sequence of rationals which can be obtained by application of the connectives. So we start by writing $\mathbf{1}$ as an abbreviation for $\neg(\varphi \dot{-} \varphi)$, where φ is any formula, and $\mathbf{0}$ as an abbreviation for $\neg\mathbf{1}$. Then, for each $r \in [0,1] \cap \mathbb{Q}$, we may introduce a unary connective r and stipulate that $r\mathbf{1}$ is a formula (with the obvious semantics). So doing, the notation and the formulation of the axiom system would be unnecessarily complicated. We write $\mathbf{2}^{-n}$ as an abbreviation for $\underbrace{\tfrac{1}{2}\dots\tfrac{1}{2}}_{n \text{ times}}\mathbf{1}$. Loosely speaking, by closing $\{\mathbf{1}\}$ with respect to the application of the connectives in Σ_1 we get what we may call the set $\mathcal{D}$ of *dyadic logic constants*. Formally speaking, in all structures $\langle M, r, v\rangle$, for all $m \in M$ we have $v(m, \mathcal{D}) = \mathbb{D}$, where $\mathbb{D}$ is the set of dyadic rationals in $[0,1]$. As in [3], for simplicity of notation from now on we identify the set $\mathcal{D}$ with its valuation $\mathbb{D}$ and we avoid the use of boldface constants.

Finally, we recall that $\mathbb{D}$ is a dense subset of the real unit interval. Hence every real in $[0,1]$ is the limit of a sequence of dyadic rationals. We may achieve an analogous result by using, for instance, the connective $\frac{1}{3}$ (indeed every real in the unit interval has a ternary expansion), but the axioms corresponding to A5 and A6 below would not be equally simple.

We comment about the semantics of $\Box$ in the following remark.

Remark 1. *By letting* $R = \{(m,k) \in M^2 : \ r(m,k) = 0\}$, *we may equivalently define*

$$v(m, \Box\varphi) = \sup\{v(k,\varphi) : \ (m,k) \in R\},$$

with the convention that $\sup \varnothing = 0$. *Hence a formula* $\Box\varphi$ *is absolutely true at a world* m *exactly when* φ *is absolutely true at all worlds that are accessible from* m *or there is no accessible world from* m.

We shall write $\varphi^M(m)$ for $v(m,\varphi)$. If $\Delta \subseteq F$, we also write

(1) $M, m \vDash \Delta$ if $\delta^M(m) = 0$ for all $\delta \in \Delta$;
(2) $M \vDash \Delta$ if $M, m \vDash \Delta$ for all $m \in M$;
(3) $\Delta \vDash \varphi$ if, for all M and all $m \in M$, $M, m \vDash \Delta$ implies $M, m \vDash \varphi$.

If there exist a structure M and $m \in M$ such that $M, m \vDash \Delta$, we say that Δ is satisfiable. If $\Delta \vDash \varphi$ we say that φ is a logical consequence of Δ. Notice that we are dealing with a local notion of logical consequence.

Notice also that every CL–formula can be assigned a truth value in a CL–structure. (The accessibility relation plays no role here.) If $M = \{m\}$, we shall write φ^M for $\varphi^M(m)$. We shall denote by $\vDash_L$ and by $\vDash_{\mathrm{CL}}$ the logical consequence relations in Łukasiewicz and in Continuous Logic respectively.

3. Axioms, rules and validity

From now on, the abbreviations and the notational conventions introduced in Section 2 are in force, specifically those related to the connective $\frac{1}{2}$ and the dyadic logic constants.

The first four axioms in the list below yield an axiomatization of Łukasiewicz $[0,1]$–valued Logic:

A1. $(\varphi \dot{-} \psi) \dot{-} \varphi$

A2. $((\eta \dot{-} \varphi) \dot{-} (\eta \dot{-} \psi)) \dot{-} (\psi \dot{-} \varphi)$
A3. $(\varphi \dot{-} (\varphi \dot{-} \psi)) \dot{-} (\psi \dot{-} (\psi \dot{-} \varphi))$
A4. $(\varphi \dot{-} \psi) \dot{-} (\neg\psi \dot{-} \neg\varphi)$

The next set of axioms describes the behavior of the connective $\frac{1}{2}$.

A5. $\frac{1}{2}\varphi \dot{-} (\varphi \dot{-} \frac{1}{2}\varphi)$
A6. $\varphi \dot{-} \frac{1}{2}\varphi \dot{-} \frac{1}{2}\varphi$

Axioms A1–A6 provide the axiomatization of propositional Continuos Logic presented in [3].

In order to clarify the meaning of A5 and A6, it is convenient to introduce some derived connectives. Let us define $\dot{+}$ as follows: $\varphi \dot{+} \psi := \neg(\neg\varphi \dot{-} \psi)$. Its semantics is the following: $r \dot{+} s = \min(r + s, 1)$. Conjunction can also be defined in terms of the connectives in Σ_1. Let us write $\varphi = \psi$ as an abbreviation for $(\varphi \dot{-} \psi) \wedge (\psi \dot{-} \varphi)$. Finally, it is just a matter of "algebraic" manipulations to rewrite axioms A5 and A6 as follows: $\frac{1}{2}\varphi \dot{+} \frac{1}{2}\varphi = \varphi$. Therefore the meaning of A5 and A6 is that $\frac{1}{2}$ really behaves as expected.

The modal axioms come next:

A7. $(\Box\varphi \dot{-} \Box\psi) \dot{-} \Box(\varphi \dot{-} \psi)$
A8. $\Box(\varphi \dot{-} d) \dot{-} (\Box\varphi \dot{-} d)$ for all $d \in \mathbb{D}$.
A9. $(\Box\varphi \dot{-} d) \dot{-} \Box(\varphi \dot{-} d)$ for all $d \in \mathbb{D}$.

Axiom A7 is just a reformulation in the continuous setting of the characteristic axiom of modal logic **K**. (see [5]). So it is appropriate to say that we are dealing with a continuous version of **K**.

The meaning of A8 and A9 is that, for all $d \in \mathbb{D}$ and all formulas φ, the formulas $\Box(\varphi \dot{-} d)$ and $\Box\varphi \dot{-} d$ are provably equivalent. (Easy calculations show that both schemata are valid.) Since d ranges over $\mathbb{D}$, A8 and A9 introduce a countable family of schemata. It is an open problem whether they can be replaced by finitely many schemata.

Finally, the inference rules are:

$$\frac{\varphi \qquad \psi \dot{-} \varphi}{\psi}\ \text{MP} \qquad\qquad \frac{\vdash \varphi}{\vdash \Box\varphi}\ \text{N}$$

Derivations are defined in the usual manner. Notice that application of N rule is restricted to the case when a proof of φ does not depend on any extralogical assumption.

Let $\Delta \subseteq F$ and $\varphi \in F$. We write $\Delta \vdash \varphi$ if there exists a derivation of φ whose set of assumptions is included in Δ. In such case we say, as usual, that φ is provable from Δ.

We shall refer to the logical system just described as to CML (shortening for *Continuous Modal Logic*). We will discuss whether CML is a conservative extension of Łukasiewicz Logic and of Continuos Logic in Section 4, after establishing our main result.

We shall denote by $\vdash_{Ł}$ and $\vdash_{CL}$ provability in Łukasiewicz Logic and in Continuous Logic respectively.

Remark 2. *Let $\varphi(P_1,\dots,P_n)$ be a formula of Continuous Logic whose proposition letters are among those displayed. Suppose $\vdash_{CL} \varphi(P_1,\dots,P_n)$. As CML extends Continuous Logic, by substitution we get $\vdash \varphi(\psi_1,\dots,\psi_n)$, for all $\psi_1,\dots,\psi_n$ in F. An analogous consideration applies to Łukasiewicz logic. In the following we shall use these facts without further mention.*

It is a matter of easy calculation to check that, for $f,g : M \to [0,1]$,

$$\sup_{k\in M} f(k) \dot{-} \sup_{k\in M} g(k) \leq \sup_{k\in M}(f \dot{-} g)(k).$$

Also, for all $m,k \in M$ and all $r : M\times M \to \{0,1\}$, we have

$$(f(k) \dot{-} r(m,k)) \dot{-} (g(k) \dot{-} r(m,k)) = (f(k) \dot{-} g(k)) \dot{-} r(m,k)$$

Therefore we get at once the following

Lemma 3. *Let M be a structure. Then, for all $m \in M$,*

$$(\Box\varphi \dot{-} \Box\psi)^M(m) \leq (\Box(\varphi \dot{-} \psi))^M(m).$$

Remark 4. *The previous lemma does non hold for arbitrary $r : M \times M \to [0,1]$. A counterexample is as follows: let M be any set and let P,Q be distinct proposition letters. Let $v(m,P) = 1$, $v(m,Q) = 1/2$ and $r(m,k) = 1/2$ for all $m,k \in M$.*

Proposition 5. *All the axioms are valid with respect to the CM semantics.*

Proof. Axioms A1,...,A6 are valid because CML semantics extends Continuous Logic. Lemma 3 states validity of A7. Finally, easy calculations show that the remaining axioms are valid. □

Proposition 6. *(Soundness) For all $\Delta \subseteq F$ and all $\varphi \in F$*

$$\Delta \vdash \varphi \;\Rightarrow\; \Delta \vDash \varphi.$$

Proof. By induction on the length of a derivation of φ from Δ, with the use of Proposition 5. Concerning the deduction rules, their validity can be easily verified. □

4. Approximated completeness

In this section, we want to establish a partial converse of Proposition 6. We will prove an *approximated* completeness theorem of the form: for all $\Delta \cup \{\varphi\} \subseteq F$

$$\Delta \vDash \varphi \;\Rightarrow\; \Delta \vdash \varphi \dot{-} 2^{-n} \quad \text{for all } n \in \omega.$$

We point out that the conclusion of the above implication cannot be strengthened. Indeed let P be a propositional variable. It can be easily verified that $\{P \dot{-} 2^{-n} : \; n \in \omega\} \vDash P$. If it were that $\{P \dot{-} 2^{-n} : \; n \in \omega\} \vdash P$ then, for some $k \in \omega$, $\{P \dot{-} 2^{-n} : \; n < k\} \vdash P$ (provability in CML is a finitary relation). Hence $\{P \dot{-} 2^{-n} : \; n < k\} \vDash P$, which is not the case.

We begin with some preliminary notions and results.

Definition 7. *Let $\Delta \subseteq F$ and let $\varphi \in F$.*

(1) Δ *is consistent if* $\Delta \nvdash \psi$ *for some* $\psi \in F$;
(2) Δ *is* φ*–consistent if* $\Delta \nvdash \varphi$.

The same definition of consistency as above is given, *mutatis mutandis*, in Łukasiewicz and in Continuous Logic. It is clear that φ–consistency implies consistency.

The following fact are thoroughly discussed and properly credited in [3].

Fact 8. *(Weak Completeness for Łukasiewicz Logic) For every formula φ of Ł–logic*

$$\vDash_{Ł} \varphi \quad \Leftrightarrow \quad \vdash_{Ł} \varphi$$

Fact 9. *Let $\Delta \subseteq F_{\mathrm{CL}}$. Then Δ is satisfiable if and only if it is consistent.*

Fact 10. *(Approximated Strong Completeness for Continuous Logic) For any $\Delta \cup \{\varphi\} \subseteq F_{\mathrm{CL}}$*

$$\Delta \vDash_{\mathrm{CL}} \varphi \quad \Leftrightarrow \quad \Delta \vdash_{\mathrm{CL}} \varphi \dot{-} 2^{-n} \text{ for all } n \in \omega.$$

Fact 11. *Let $\Delta \subseteq F_{\mathrm{CL}}$. Then Δ is inconsistent if and only if $\Delta \vdash_{\mathrm{CL}} d$, for some $d \in \mathbb{D} \setminus \{0\}$. We prove the nontrivial implication. Remember that we are identifying each dyadic logic constant with its valuation (see the above comments on the role of the connective $\frac{1}{2}$). Let $0 < d \in \mathbb{D}$ be such that $\Delta \vdash_{\mathrm{CL}} d$. By Soundness, in any model M of Δ, the rational d must take value zero, which is impossible. It follows that Δ is unsatisfiable, hence inconsistent by Fact 9.*

Fact 12. *Let $\Delta \cup \{\varphi\} \subseteq F_{\mathrm{CL}}$ be finite. Then*

$$\Delta \vDash_{\mathrm{CL}} \varphi \quad \Leftrightarrow \quad \Delta \vdash_{\mathrm{CL}} \varphi.$$

Remark 13.

(1) *$\Delta \subseteq F$ is inconsistent if and only if $\Delta \vdash d$ for some $d \in \mathbb{D} \setminus \{0\}$: for the "if" part, suppose $\Delta \vdash d$, for some positive dyadic d. As $d \vdash_{\mathrm{CL}} P$ for any propositional letter P, by substitution we get $\Delta \vdash \varphi$ for all $\varphi \in F$.*
(2) *Every φ–consistent set extends to a maximal φ–consistent set, by a standard application of Zorn's Lemma.*
(3) *Suppose Δ is maximal φ–consistent and $\Delta \vdash \psi$ Then $\Delta, \psi \nvdash \varphi$. By maximality, $\psi \in \Delta$. Hence Δ is closed under provability.*

We recursively define $\varphi \dot{-} n\psi$, $n \in \omega$, as follows:

$$\varphi \dot{-} 0\psi = \varphi; \qquad \varphi \dot{-} (n+1)\psi = (\varphi \dot{-} n\psi) \dot{-} \psi.$$

Same as in [3], We have the following

Theorem 14. *(Deduction Theorem) Let $\Delta \subseteq F$ and let $\varphi, \psi \in F$. Then $\Delta, \psi \vdash \varphi$ if and only if $\Delta \vdash \varphi \dot{-} n\psi$ for some $n \in \omega$.*

Proof. We repeat the proof of [3, Theorem 8.1]. Concerning the non-trivial implication, first of all we observe that, for all α, β, γ and and all $n, m \in \omega$,

$$\vdash_L ((\beta \dot{-} (n+m)\alpha) \dot{-} ((\beta \dot{-} \gamma) \dot{-} n\alpha)) \dot{-} (\gamma \dot{-} m\alpha).$$

This can be shown using completeness of Łukasiewicz propositional logic with respect to the MV-algebra $[0,1]$.

The proof then proceeds by induction on a derivation of φ from $\Delta \cup \{\psi\}$. Notice that the presence of additional axioms to those in [3] and of N rule poses non problem. Indeed the N rule can only be applied when the set premises is empty.

□

Proposition 15. *Let $\Delta \subseteq F$ and let $\varphi, \psi, \eta \in F$. If $\Delta, \varphi \dot{-} \psi \vdash \eta$ and $\Delta, \psi \dot{-} \varphi \vdash \eta$ then $\Delta \vdash \eta$.*

Proof. By the Deduction Theorem, there exist $m, n \in \omega$ such that $\Delta \vdash \eta \dot{-} m(\varphi \dot{-} \psi)$ and $\Delta \vdash \eta \dot{-} n(\psi \dot{-} \varphi)$. Notice that

$$\vdash_{\text{Ł}} P \dot{-} (P \dot{-} m(Q \dot{-} R)) \dot{-} (P \dot{-} n(R \dot{-} Q))$$

for all proposition letters P, Q, R and all $m, n \in \omega$. Hence the conclusion follows by a double application of MP. □

Now we discuss properties of maximal φ–consistent sets, where φ is an arbitrary formula.

Proposition 16. *Let $\Delta \subseteq F$ be maximal φ–consistent, For all $\psi, \eta \in \Delta$ at least one of $\psi \dot{-} \eta$ and $\eta \dot{-} \psi$ is in Δ.*

Proof. By contraposition. Suppose $\psi \dot{-} \eta \notin \Delta$ and $\eta \dot{-} \psi \notin \Delta$, for some ψ, η. Then, by maximality, $\Delta, \psi \dot{-} \eta \vdash \varphi$ and $\Delta, \eta \dot{-} \psi \vdash \varphi$. By Proposition 15 we get $\Delta \vdash \varphi$.

□

Theorem 17. *Let $\Delta \subseteq F$ be maximal φ–consistent, Then, for every $\psi \in F$,*

$$\sup\{s \in \mathbb{D} :\ s \dot{-} \psi \in \Delta\} = \inf\{s \in \mathbb{D} :\ \psi \dot{-} s \in \Delta\}.$$

Proof. If the left–hand side were smaller than the right–hand side, pick $s \in \mathbb{D}$ strictly between sup and inf . By Proposition 16, at least one of $s \dot{-} \psi, \psi \dot{-} s$ is in Δ, but $s \dot{-} \psi \in \Delta$ contradicts to the sup being smaller than s and $\psi \dot{-} s \in \Delta$ contradicts to the inf being greater than s.

Furthermore, the left-hand side cannot be greater than the right-hand side, otherwise $s \dot{-} \psi$ and $\psi \dot{-} t$ both belong to Δ for some $s, t \in \mathbb{D}$ such that $s > t$. Then, by axiom A2, $\Delta \vdash (s \dot{-} t)$. Hence Δ would be inconsistent (see Fact 11), contrary to our assumption.

□

Let Δ be maximal φ–consistent and let $\psi \in F$. According to Theorem 17, we let

$$\psi^{\Delta} = \sup\{s \in \mathbb{D} :\ s \dot{-} \psi \in \Delta\} = \inf\{s \in \mathbb{D} :\ \psi \dot{-} s \in \Delta\}.$$

Next we take inspiration from Fitting's construction of a canonical model for a many-valued modal logic (see [10]) and we define a *canonical structure* $\langle M, r, v\rangle$, where

(1) elements of M are maximal φ–consistent sets, for some $\varphi \in F$;
(2) $r(\Delta, \Gamma) = 0$ if and only if $\psi^{\Gamma} \leq (\Box\psi)^{\Delta}$, for all $\psi \in F$;
(3) $v(\Delta, P) = P^{\Delta}$ for all $\Delta \in M$ and all proposition letters P.

Notice that condition (2) above, in classical modal logic translates into

$$\Delta\, R\, \Gamma \ \Leftrightarrow\ \text{for all formulas } \psi\ (\Box\psi \in \Delta\ \Rightarrow\ \psi \in \Gamma),$$

which is the standard requirement on the accessibility relation R when constructing a canonical model.

Theorem 18. *For all $\psi \in F$ and all $\Delta \in M$, $\psi^M(\Delta) = \psi^{\Delta}$.*

Proof. By induction on ψ.

(a) For atomic ψ the conclusion holds by condition (3) above.
(b) Let ψ be of the form $\frac{1}{2}\eta$. Let us assume $\eta^M(\Delta) = \eta^{\Delta}$. Then

$$\psi^M(\Delta) = \frac{1}{2}\eta^M(\Delta) = \frac{1}{2}\eta^{\Delta} = \frac{1}{2}\sup\{d \in \mathbb{D} :\ (d \dot{-} \eta) \in \Delta\} = \sup\{\frac{1}{2}d : (d \dot{-} \eta) \in \Delta\}.$$

As Δ is is closed under provability and $(d \dot- \eta)$ is provably equivalent to $(\frac{1}{2}d \dot- \frac{1}{2}\eta)$ in Continuous Logic, the last term in the above chain of equalities is just $(\frac{1}{2}\eta)^\Delta$.

(c) Let ψ be of the form $\neg\eta$. Let us assume $\eta^M(\Delta) = \eta^\Delta$. Then

$$\psi^M(\Delta) = 1 - \eta^M(\Delta) = 1 - \eta^\Delta = 1 - \sup\{d \in \mathbb{D} : \ (d \dot- \eta) \in \Delta\}.$$

Since $\alpha \dot- \beta$ and $\neg\beta \dot- \neg\alpha$ are provably equivalent in Łukasiewicz logic for all α, β, it follows from closure under provability of Δ that $(d \dot- \eta) \in \Delta$ if and only if $(\neg\eta \dot- (1-d)) \in \Delta$. So the last term in the above chain of equalities is equal to $\inf\{1 - d : (\neg\eta \dot- (1-d)) \in \Delta\} = (\neg\eta)^\Delta$.

(d) Let ψ be of the form $\eta \dot- \xi$. Let us assume $\eta^M(\Delta) = \eta^\Delta$ and $\xi^M(\Delta) = \xi^\Delta$. Then $(\eta \dot- \xi)^M = \eta^\Delta \dot- \xi^\Delta = \sup\{d : \ d \dot- \eta \in \Delta\} \dot- \inf\{e : \xi \dot- e \in \Delta\} = \sup\{d \dot- e : d \dot- \eta \in \Delta \text{ and } \xi \dot- e \in \Delta\} \leq \sup\{d : d \dot- (\eta \dot- \xi) \in \Delta\} = (\eta \dot- \xi)^\Delta$, where, for the last inequality, we notice that

$$d \dot- P, Q \dot- e \vDash_{\mathrm{CL}} (d \dot- e) \dot- (P \dot- Q),$$

for all propositional letters P, Q. Hence, by Fact 12 and by Remark 2, we get

$$d \dot- \eta \in \Delta \text{ and } \xi \dot- e \in \Delta \ \Rightarrow \big((d \dot- e) \dot- (\eta \dot- \xi)\big) \in \Delta.$$

Similarly, $\eta^\Delta \dot- \xi^\Delta = \inf\{d : \ (\eta \dot- d) \in \Delta\} \dot- \sup\{e : \ (e \dot- \xi) \in \Delta\} = \inf\{d \dot- e : \ (\eta \dot- d) \in \Delta \text{ and } (e \dot- \xi) \in \Delta\} \geq \inf\{d : \ \big(d \dot- (\eta \dot- \xi)\big) \in \Delta\} = (\eta \dot- \xi)^\Delta$, where the last inequality follows form the above implication.

(e) Let ψ be of the form $\Box\eta$.

In the following we repeatedly use that $\vdash_{\mathrm{CL}} (P \dot- d_1) \dot- (P \dot- d_2)$ for all proposition letters P and all $d_1, d_2 \in \mathbb{D}$ such that $d_2 \leq d_1$. (To show this, use Fact 12). Hence, for all $\chi \in F$ and all $d_2 \leq d_1$ in $\mathbb{D}$,

$$\vdash (\chi \dot- d_1) \dot- (\chi \dot- d_2).$$

Let us inductively assume that $\eta^M(\Delta) = \eta^\Delta$ for all $\Delta \in M$. Let $\Delta \in M$ be φ–consistent. First of all, recalling Remark 1 and condition (2) above, we have that

$$(\Box\eta)^M(\Delta) = \sup_{\Delta R \Gamma} \eta^M(\Gamma) = \sup_{\Delta R \Gamma} \eta^\Gamma \leq (\Box\eta)^\Delta.$$

For sake of contradiction, assume that $(\Box\eta)^M(\Delta) < (\Box\eta)^\Delta$. Let $\bar{d} \in \mathbb{D}$ be such that $(\Box\eta)^M(\Delta) < \bar{d} < (\Box\eta)^\Delta$. Let

$$\Delta' = \bigcup_{\xi \in F} \{\xi \dot- d : \ d \in \mathbb{D} \text{ and } d > (\Box\xi)^\Delta\}.$$

We claim that $\Delta' \nvdash \eta \dot{-} \bar{d}$. Otherwise there exist k, $\xi_i \in F$ and $d_i > (\Box\xi_i)^{\Delta}$, $1 \leq i \leq k$, such that

$$\xi_1 \dot{-} d_1, \ldots, \xi_k \dot{-} d_k \vdash \eta \dot{-} \bar{d}.$$

By repeated applications of the Deduction Theorem, we get

$$\vdash (\eta \dot{-} \bar{d}) \dot{-} n_1(\xi_1 \dot{-} d_1) \dot{-} \ldots \dot{-} n_k(\xi_k \dot{-} d_k),$$

for some $n_1, \ldots, n_k \in \omega$.

Application of N rule yields

$$\vdash \Box\big((\eta \dot{-} \bar{d}) \dot{-} n_1(\xi_1 \dot{-} d_1) \dot{-} \ldots \dot{-} n_k(\xi_k \dot{-} d_k)\big).$$

By multiple applications of axioms A7, A8, A9 and of MP, we get

$$\Box\xi_1 \dot{-} d_1, \ldots, \Box\xi_k \dot{-} d_k \vdash \Box\eta \dot{-} \bar{d}.$$

Since $(\Box\xi_i \dot{-} d_i) \in \Delta$ for all $1 \leq i \leq k$, we get that $\Delta \vdash \Box\eta \dot{-} \bar{d}$. So, by Remark 13 (3), $(\Box\eta \dot{-} \bar{d}) \in \Delta$, contradicting to our choice of $\bar{d}$. Therefore $\Delta' \nvdash \eta \dot{-} \bar{d}$.

Let Σ be a maximal $(\eta \dot{-} \bar{d})$–consistent extension of Δ'. By definition of Δ' we have that, for every $\xi \in F$, $\xi^{\Sigma} \leq (\Box\xi)^{\Delta}$. Hence $\Delta \, R \, \Sigma$.

By construction, $\eta \dot{-} \bar{d} \notin \Sigma$, so $\eta^{\Sigma} > (\Box\eta)^M(\Delta)$. On the other hand,

$$(\Box\eta)^M(\Delta) = \sup_{\Delta R \Gamma} \eta^M(\Gamma) = \sup_{\Delta R \Gamma} \eta^{\Gamma} \geq \eta^{\Sigma},$$

hence a contradiction. Therefore $(\Box\eta)^M(\Delta) = (\Box\eta)^{\Delta}$ for all $\Delta \in M$.

□

Corollary 19. *The following are equivalent for all $\Delta \subseteq F$.*

(1) Δ *is satisfiable;*
(2) Δ *is consistent.*

Proof. As for the nontrivial implication, let Γ be a maximal consistent extension of a consistent Δ. Then, by Theorem 18 above, we have that, with respect to the canonical structure M previously defined,

$$0 = \delta^{\Gamma} = \delta^M(\Gamma) \quad \text{for all } \delta \in \Delta.$$

Hence Δ is satisfiable. □

Since the provability relation is finitary, from the previous corollary we easily get the following:

Corollary 20. *(Compactness of CML) A set of formulas is satisfiable if and only if it is finitely satisfiable.*

Corollary 21. *(Approximated Completeness Theorem for CML) For all* $\Delta \cup \{\varphi\} \subseteq F$

$$\Delta \vDash \varphi \quad \Leftrightarrow \quad \Delta \vdash \varphi \dot{-} 2^{-n} \textit{ for all } n \in \omega.$$

Proof. As for the nontrivial implication, suppose $\Delta \nvdash \varphi \dot{-} 2^{-n}$, for some $n \in \omega$. Let Γ be a maximal $(\varphi \dot{-} 2^{-n})$–consistent extension of Δ. By Theorem 18, in the canonical structure M it holds that $\varphi^M(\Gamma) = \varphi^\Gamma \geq 2^{-n}$. Since $\delta^M(\Gamma) = 0$ for all $\delta \in \Delta$, we get $\Delta \nvDash \varphi$. □

By importing the modal terminology into the current setting, we have that $\vDash \varphi$ means that φ is valid with respect to the class of crisp frames (in case of provability from the empty set of assumptions, the global and the local logical consequence relation are the same). From Corollary 21 we derive that the validity of φ with respect to the class of crisp frames is in turn equivalent to φ being an approximated logical theorem, where the attribute approximated is explained by the right-hand side of the equivalence in Corollary 21.

It is an open question whether Corollary 21 can be strengthened by replacing the right–hand side of the equivalence with $\Delta \vdash \varphi$, when Δ is finite. (Compare with Fact 12.) Notice that the canonical model construction does not provide an answer at once.

Remark 22. *CML is a conservative extension of Łukasiewicz logic in the following sense: for every formula* φ *in the language of Łukasiewicz logic it holds that*

$$\vdash \varphi \;\Leftrightarrow\; \vdash_L \varphi.$$

Concerning the left-to-right implication, we have $\vdash \varphi \;\Rightarrow\; \vDash \varphi$*, by the Soundness Theorem. A fortiori,* φ *is valid in the MV-algebra* $[0,1]$ *under all assignments of values to its propositional variables. Hence, by weak completeness of Łukasiewicz logic with respect to the* $[0,1]$*-algebra, we conclude that* $\vdash_L \varphi$*.*

We can also establish a conservativity result of CML relative to CL. The following hold for every formula φ *in the language of CL:*

$$\vdash \varphi \dot{-} 2^{-n} \;\textit{ for all } n \in \omega \iff \;\vdash_{\mathrm{CL}} \varphi \dot{-} 2^{-n} \;\textit{ for all } n \in \omega.$$

As for the left-to-right implication, let us denote by $\vDash_{\mathrm{CL}}$ *the logical consequence relation in CL. By Corollary 21,* $\vdash \varphi \dot{-} 2^{-n}$ *for all* $n \in \omega \Rightarrow \vDash \varphi$*. A fortiori,* $\vDash_{\mathrm{CL}} \varphi$ *holds. By the Theorem of Approximated Completeness for CL (Fact 10 above), we get the conclusion.*

Next we make a very final comment. We have shown in Remark 4 that the characteristic axiom of modal logic **K** becomes an unsound formula if allow for $[0,1]$–valued accessibility relations. Let us do it anyway. First of all we notice that, for all structures $\langle M, r : M \times M \to [0,1], v : M \times \Sigma_0 \to [0,1]\rangle$ and all $m \in M$, it holds that

$$\begin{aligned} r(m,m) = 0 &\Rightarrow \forall\varphi\,(M, m \vDash \varphi \dot{-} \Box\varphi); \\ \forall k, n \in M(r(m,n) \leq r(m,k) + r(k,n)) &\Rightarrow \forall\varphi\,(M, m \vDash \Box\,\Box\,\varphi \dot{-} \Box\varphi); \\ \forall k \in M(r(m,k) \leq r(k,m)) &\Rightarrow \forall\varphi\,(M, m \vDash \Box\,\Diamond\,\varphi \dot{-} \varphi). \end{aligned}$$

We say that the accessibility relation r is *symmetric* if whenever $r(m,k) = 0$ then $r(k,m) = 0$. We define *reflexivity* and *transitivity* as expected.

In addition to r being $[0,1]$–valued, let us also assume that M is *full*, i.e. for every function $f : M \to [0,1]$ there exists a proposition symbol P such that $P^M = f$. (Admittedly a strong assumption.) Then all the above implications are indeed equivalences. It follows that

$$\begin{aligned} \forall\varphi\,(M \vDash \varphi \dot{-} \Box\varphi) &\Rightarrow r \text{ is reflexive;} \\ \forall\varphi\,(M \vDash \Box\,\Diamond\,\varphi \dot{-} \varphi) &\Rightarrow r \text{ is symmetric;} \\ \forall\varphi\,(M \vDash \Box\,\Box\,\varphi \dot{-} \Box\varphi) &\Rightarrow r \text{ is transitive,} \end{aligned}$$

Finally notice that the last three implications are reminiscent of modal logic **S5**.

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