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THE FULL DIRAC OPERATOR ON A CLIFFORD ALGEBRA

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Abstract. *We present some properties of a first order differential operator $\mathcal{D}_n$ on the real Clifford algebra $\mathbb{R}_{0,n}$. The operator $\mathcal{D}_n$ extends the Dirac and Weyl operators to functions that can depend on all the coordinates of the algebra. Our starting point is a modified Cauchy-Riemann-Fueter operator on the quaternions, which is the sum of two one-variable Cauchy-Riemann operators. The operator $\mathcal{D}_n$ behaves well both w.r.t. monogenic functions and w.r.t. the powers of the (complete) Clifford variable x. This last property relates the operator $\mathcal{D}_n$ with the recent theory of slice monogenic and slice regular functions.*

1 INTRODUCTION

In this paper we study some basic properties of a first order differential operator on the real Clifford algebra $\mathbb{R}_n := \mathbb{R}_{0,n}$ which generalizes the Weyl operator used in the theory of monogenic functions (for which we refer to [1], [2], [5]). While monogenic functions are usually defined on open subsets of the paravector space, the operator we consider acts on functions that can depend on all the coordinates of the algebra. This is similar to what happens on the quaternionic space $\mathbb{H} \simeq \mathbb{R}_2$, where the Cauchy-Riemann-Fueter operator acts on the whole space, not only on the reduced quaternions $\mathbb{H}_3 = \langle 1, i, j \rangle$. Our starting point is the *modified* Cauchy-Riemann-Fueter operator on the quaternions:

$$\mathcal{D} = \frac{1}{2}\left(\partial_{x_0} + i\partial_{x_1} + j\partial_{x_2} - k\partial_{x_3}\right) \tag{1}$$

(see [12] and [9] for some properties of this and related operators). When written in the notation of the Clifford algebra $\mathbb{R}_2$, it becomes the operator

$$\mathcal{D}_2 = \frac{1}{2}\left(\partial_{x_0} + e_1\partial_{x_1} + e_2\partial_{x_2} - e_{12}\partial_{x_{12}}\right). \tag{2}$$

If $\mathcal{D}_1 = \partial_{\bar{z}_1} = \frac{1}{2}\left(\partial_{x_0} + e_1\partial_{x_1}\right)$ and $\mathcal{D}_{1,2} = \frac{1}{2}\left(\partial_{x_2} + e_1\partial_{x_{12}}\right)$ are the one-variable Cauchy–Riemann operators w.r.t. the complex variables $z_1 = x_0 + e_1x_1$, $z_2 = x_2 + e_1x_{12}$, then $\mathcal{D}_2 = \mathcal{D}_1 + e_2\mathcal{D}_{1,2}$. This observation suggests a recursive definition of a differential operator $\mathcal{D}_n$ on $\mathbb{R}_n$. Even if the definition of $\mathcal{D}_n$ is not symmetric w.r.t. the basis vectors, the operator we obtain *is* symmetric, and has the following explicit form:

$$\mathcal{D}_n = \frac{1}{2}\sum_K e_K^*\partial_{x_K} \tag{3}$$

where $e_K^* = (-1)^{\frac{k(k-1)}{2}}e_K$ is obtained by applying to e_K the reversion anti-involution.

When restricted to functions of paravector variable $(x_0, x_1, \dots, x_n)$, $\mathcal{D}_n$ is equal (up to a factor $1/2$) to the Weyl (cf. e.g. [2, §4.2]), or Cauchy-Riemann (as in [5, §5.3]) operator of $\mathbb{R}_n$. Therefore every $\mathbb{R}_n$-valued monogenic function defined on an open domain of $\mathbb{R}^{n+1} \subset \mathbb{R}_n$ is in the kernel of $\mathcal{D}_n$. Moreover, the identity function x of $\mathbb{R}_n$ is in the kernel of $\mathcal{D}_n$, while its restriction to the paravector variable is *not* monogenic.

The operator $\mathcal{D}_n$ behaves well also w.r.t. powers of the (complete) Clifford variable x. We show that every power x^m is in the kernel of $\mathcal{D}_n$ when n is odd. For even n, the same property holds on the so-called *quadratic cone* of the algebra (cf. [7]). These properties link the operators $\mathcal{D}_n$ to the recent theory of *slice monogenic* [3] and *slice regular* functions on $\mathbb{R}_n$ [6, 7].

Operators similar to $\mathcal{D}_n$ have already been considered in the literature (e.g. in [10], [4] and [11]). However, it seems that the operators $\mathcal{D}_n$ are particularly well adapted to the theory of polynomials $\sum_k x^k a_k$ or more generally of slice regular functions on a Clifford algebra.

On the negative side, the operator $\mathcal{D}_n$ is not elliptic for $n > 2$ and its kernel is very large if we do not restrict the domains where functions are defined. In the last section, we focus on the case $n = 3$ and show a more strict relation of $\mathcal{D}_3$ with the Weyl operator. This suggests to consider a proper subspace of the kernel of $\mathcal{D}_3$, where the condition of *Cliffordian holomorphicity* [8] has a role. We get in this way the real analyticity in $\mathbb{R}_3$ and an integral representation formula on domains of polydisc type.

2 THE FULL DIRAC OPERATORS

Denote by $e_1, \dots, e_n$ the generators of $\mathbb{R}_n$. Let $x = \sum_K x_K e_K \in \mathbb{R}_n$, where $K = (i_1, \dots, i_k)$ is a multiindex, with $0 \le |K| := k \le n$, $x_K \in \mathbb{R}$, $e_K = e_{i_1} \cdots e_{i_k}$.

Definition 1. Let $\mathcal{D}_1 = \frac{1}{2}(\partial_{x_0} + e_1 \partial_{x_1})$ and $\mathcal{D}_{1,2} = \frac{1}{2}(\partial_{x_2} + e_1 \partial_{x_{12}})$. For $n > 1$, define recursively

$$\mathcal{D}_n := \mathcal{D}_{n-1} + e_n \mathcal{D}_{n-1,n} \tag{4}$$

where we consider $\mathbb{R}_{n-1}$ embedded in $\mathbb{R}_n$ and $\mathcal{D}_{n-1,n}$ is the operator defined as $\mathcal{D}_{n-1}$ w.r.t. the 2^{n-1} variables $x_n, x_{1n}, x_{2n}, \dots, x_{12n}, \dots, x_{12\cdots n}$. Since D_n depends on all the basis coordinates of $\mathbb{R}_n$, we call it the *full Dirac operator* on $\mathbb{R}_n$.

Remark 1. The operator $\mathcal{D}_1$ is the standard Cauchy–Riemann operator on $\mathbb{R}_1 \simeq \mathbb{C}$. $\mathcal{D}_2$ is the modified Cauchy-Riemann-Fueter operator on $\mathbb{R}_2 \simeq \mathbb{H}$:

$$\mathcal{D}_2 = \frac{1}{2}\left(\partial_{x_0} + e_1 \partial_{x_1} + e_2 \partial_{x_2} - e_{12} \partial_{x_{12}}\right). \tag{5}$$

$\mathcal{D}_3 = \mathcal{D}_2 + e_3 \mathcal{D}_{2,3}$ is the following operator

$$\mathcal{D}_3 = \frac{1}{2}\left(\partial_{x_0} + e_1 \partial_{x_1} + e_2 \partial_{x_2} + e_3 \partial_{x_3} - e_{12} \partial_{x_{12}} - e_{13} \partial_{x_{13}} - e_{23} \partial_{x_{23}} - e_{123} \partial_{x_{123}}\right).$$

An easy inductive procedure shows that, despite its recursive definition, the operator $\mathcal{D}_n$ is symmetric w.r.t. the basis elements $e_1, \dots, e_n$.

Proposition 1. *The operator $\mathcal{D}_n$ can be written in the following form:*

$$\mathcal{D}_n = \frac{1}{2} \sum_{|K| \le n} e_K^* \partial_{x_K} \tag{6}$$

where $e_K^* = (-1)^{\frac{k(k-1)}{2}} e_K$ is obtained by applying to e_K the reversion *anti-involution $x \mapsto x^*$. Moreover, $\mathcal{D}_{n-1,n} = \frac{1}{2} \sum_{H \not\ni n} e_H^* \partial_{x_{(Hn)}}$.* □

On functions depending only on paravectors, the operator $\mathcal{D}_n$ is equal to the $\frac{1}{2}\mathcal{W}_n$, where $\mathcal{W}_n$ is the Weyl operator $\mathcal{W}_n = \partial_{x_0} + \sum_{i=1}^n e_i \partial_{x_i}$.

Corollary 2. *Every monogenic function (i.e. in the kernel of $\mathcal{W}_n$) defined on an open subset of the paravector subspace $\mathbb{R}^{n+1} \subset \mathbb{R}_n$ can be identified with an element of* $\ker \mathcal{D}_n$. □

We can define also the conjugated operator $\overline{\mathcal{D}}_n$ and the auxiliary operator $\mathcal{D}_n^*$.

Definition 2.

$$\begin{cases} \overline{\mathcal{D}}_1 = \partial_{z_1} = \frac{1}{2}(\partial_{x_0} - e_1 \partial_{x_1}), \quad \mathcal{D}_1^* = \partial_{\bar{z}_1} = \frac{1}{2}(\partial_{x_0} + e_1 \partial_{x_1}) \\ \overline{\mathcal{D}}_n = \overline{\mathcal{D}}_{n-1} - e_n \mathcal{D}_{n-1,n}^*, \quad \mathcal{D}_n^* = \mathcal{D}_{n-1}^* + e_n \overline{\mathcal{D}}_{n-1,n} \end{cases}$$

where $\mathcal{D}_{n-1,n}^*$ and $\overline{\mathcal{D}}_{n-1,n}$ are defined as $\mathcal{D}_{n-1}^*$ and $\overline{\mathcal{D}}_{n-1}$ w.r.t. the 2^{n-1} variables $x_n, x_{1n}, \dots,$ $x_{12n}, \dots, x_{12\cdots n}$.

The operators $\overline{\mathcal{D}}_n$ and $\mathcal{D}_n^*$ have the following explicit forms:

Proposition 3.

$$\overline{\mathcal{D}}_n = \frac{1}{2}\sum_{|K|\le n} \tilde{e}_K \partial_{x_K} \tag{7}$$

where $\tilde{e}_K = (-1)^k e_K$ *is obtained by applying to* e_K *the* principal involution $x \mapsto \tilde{x}$. *Moreover,*

$$\mathcal{D}_n^* = \frac{1}{2}\sum_{|K|\le n} e_K \partial_{x_K}, \quad \mathcal{D}_{n-1,n}^* = \frac{1}{2}\sum_{H \not\ni n} e_H \partial_{x_{Hn}} \tag{8}$$

□

Remark 2. The identity function x of $\mathbb{R}_3$ is in the kernel of $\mathcal{D}_2$ and $\mathcal{D}_3$ (and of course of $\mathcal{D}_1$). Starting from $\mathcal{D}_1 x = 0$, $\mathcal{D}_{1,2} x = 0$ on $\mathbb{R}_2$, we get recursively that $\mathcal{D}_n x = 0$ on $\mathbb{R}_n$ for every n. Note that even if $\mathcal{D}_1^* x = \mathcal{D}_{1,2}^* x = 0$, the identity function does not belong to the kernels of $\mathcal{D}_n^*$ or $\overline{\mathcal{D}}_n$ for every n.

The operator $\mathcal{D}_n^*$ has already been considered in the literature (cf. [10] and [4]). The property given in the preceding remark and the behavior of $\mathcal{D}_n$ w.r.t. power functions (see Theorem (5)) indicate that these operators are better suited than $\mathcal{D}_n^*$ or $\overline{D}_n$ to the theory of polynomials or more generally *slice regular* functions on a Clifford algebra. In [11] Dirac operators on the subspace of l-vectors have been studied. They coincide (up to sign) with the restriction of $\mathcal{D}_n$ to l-vectors. Since we are interested in the global behavior of the operator on the algebra, the choice of the grade-depending sign for the coefficients of $\mathcal{D}_n$ is essential.

We are interested in the values of $\mathcal{D}_n$ on the powers of the complete Clifford variable x. To express our computation, we need some definitions and results from the theory of *slice regular* functions on $\mathbb{R}_n$ (see [6, 7]).

Definition 3. Let $t(x) = x + \bar{x}$ be the *trace* of x and $n(x) = x\bar{x}$ the *norm* of $x \in \mathbb{R}_n$. The *quadratic cone* of $\mathbb{R}_n$ is the subset

$$\mathcal{Q}_n := \mathbb{R} \cup \{x \in \mathbb{R}_n \mid t(x) \in \mathbb{R},\ n(x) \in \mathbb{R}, 4n(x) > t(x)^2\}.$$

Let $\mathbb{S}_n := \{J \in \mathcal{Q}_n \mid J^2 = -1\} = \{x \in \mathbb{R}_n \mid t(x) = 0,\ n(x) = 1\}$.

Proposition 4 ([6, 7])**.** *The quadratic cone* $\mathcal{Q}_n$ *satisfies the following properties:*

1. $\mathcal{Q}_n = \mathbb{R}_n$ *only for* $n = 1, 2$.
2. $\mathcal{Q}_n$ *contains the subspace of paravectors* $\mathbb{R}^{n+1} := \{x \in \mathbb{R}_n \mid [x]_k = 0$ *for every* $k > 1\}$.
3. $\mathcal{Q}_n$ *is the real algebraic subset (proper for* $n > 2$*) of* $\mathbb{R}_n$ *defined by the equations*

$$x_K = 0,\ x \cdot (x e_K) = 0 \quad \forall e_K \neq 1 \text{ such that } e_K^2 = 1.$$

4. *For* $J \in \mathbb{S}_n$, *let* $\mathbb{C}_J := \langle 1, J\rangle \simeq \mathbb{C}$ *be the subalgebra generated by* J. *Then* $\mathcal{Q}_n = \bigcup_{J\in\mathbb{S}_n} \mathbb{C}_J$ *and* $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$ *for every* $I, J \in \mathbb{S}_n$, $I \neq \pm J$. *As a consequence, if* x *belongs to* $\mathcal{Q}_n$, *also the powers* x^m *belong to* $\mathcal{Q}_n$.

□

Slice regular functions are defined only on subdomains of the quadratic cone (we refer to [6, 7] for full details). However, if the domain intersects the real axis, then the class of slice regular functions coincides with that of functions having local power series expansion centered at real points. Now we compute the values of $\mathcal{D}_n(x^m)$.

Theorem 5. *Let $x = \sum_{|K|\le n} x_K e_K$ denote the complete Clifford variable in $\mathbb{R}_n$.*

1. *If n is an odd integer, then $\mathcal{D}_n(x^m) = 0$ on the whole algebra $\mathbb{R}_n$ for every integer $m \ge 1$.*
2. *If n is an even integer, $n > 2$, then $\mathcal{D}_n(x^m) = 0$ on the quadratic cone $\mathcal{Q}_n$ of $\mathbb{R}_n$ for every integer $m \ge 1$.*
3. *$\mathcal{D}_2(x^m) = 0$ on the subset of reduced quaternions $\mathbb{H}_3 \subseteq \mathbb{R}_2$ for every integer $m \ge 1$.*

For the proof of the Theorem we apply the following algebraic lemma:

Lemma 6. *For every $x \in \mathbb{R}_n$, it holds*

$$\sum_{H \not\ni n} e_H^* x e_H = 2^{n-1}(x_n e_n + x_{(1\cdots n-1)} e_{(1\cdots n-1)}) \quad \text{for odd } n \tag{9}$$

$$\sum_{H \not\ni n} e_H^* x e_H = 2^{n-1}(x_n e_n + x_N e_N) \quad \text{for even } n. \tag{10}$$

where $x_N = x_{(1\cdots n)}$ and $e_N = e_{(1\cdots n)}$ is the pseudoscalar in $\mathbb{R}_n$. □

Proof of the Theorem. Case (1): n odd. We show that $\mathcal{D}_{n-1}x^m = -e_n \mathcal{D}_{n-1,n} x^m$ by induction on m. Since $\mathcal{D}_n x = 0$, the equality is valid for $m = 1$. Take $m > 1$ and assume that $\mathcal{D}_{n-1}x^{m-1} = -e_n \mathcal{D}_{n-1,n} x^{m-1}$. We have the following product formula:

$$\begin{aligned} \mathcal{D}_{n-1,n} x^m &= (\mathcal{D}_{n-1,n} x^{m-1})x + \frac{1}{2} \sum_{H \not\ni n} e_H^* x^{m-1} e_{(Hn)} && (11) \\ &= (\mathcal{D}_{n-1,n} x^{m-1})x + \frac{1}{2} \left(\sum_{H \not\ni n} e_H^* x^{m-1} e_H \right) e_n. && (12) \end{aligned}$$

Since, from Lemma 6,

$$\left(\sum_{H \not\ni n} e_H^* x e_H \right) e_n = 2^{n-1}(-x_n + x_{(1\cdots n-1)} e_N), \tag{13}$$

the last term in equation (11) belongs to the center $\langle 1, e_N \rangle$ of $\mathbb{R}_n$. Therefore, from (11) we get

$$-e_n \mathcal{D}_{n-1,n} x^m = -e_n(\mathcal{D}_{n-1,n} x^{m-1})x + \frac{1}{2} \sum_{H \not\ni n} e_H^* x^{m-1} e_H. \tag{14}$$

On the other hand, we also have

$$\mathcal{D}_{n-1} x^m = (\mathcal{D}_{n-1} x^{m-1})x + \frac{1}{2} \sum_{H \not\ni n} e_H^* x^{m-1} e_H \tag{15}$$

and then the inductive hypothesis gives the equality $\mathcal{D}_{n-1}x^m = -e_n\mathcal{D}_{n-1,n}x^m$, which is equivalent to $\mathcal{D}_n x^m = 0$.

Case (2): n even. We show that

$$\mathcal{D}_n x^m = (\mathcal{D}_n x^{m-1})x + 2^{n-1}\left[x^{m-1}\right]_N e_N \tag{16}$$

where $[a]_N$ denotes the coefficient of the pseudoscalar e_N in $a \in \mathbb{R}_n$. If $m = 1$, $\mathcal{D}_n x = 0$ and the equality 16 is true. Let $m > 1$. Then

$$\mathcal{D}_n x^m = \mathcal{D}_{n-1}x^m + e_n\mathcal{D}_{n-1,n}x^m \tag{17}$$

$$= (\mathcal{D}_{n-1}x^{m-1})x + \tfrac{1}{2}\sum_{H\not\ni n} e_H^* x^{m-1} e_H + e_n((\mathcal{D}_{n-1,n}x^{m-1})x + \tfrac{1}{2}\sum_{H\not\ni n} e_H^* x^{m-1} e_H e_n) \tag{18}$$

From Lemma 6, since n is even we have

$$\sum_{H\not\ni n} e_H^* x^{m-1} e_H + e_n \sum_{H\not\ni n} e_H^* x^{m-1} e_H e_n = 2^n\left[x^{m-1}\right]_N e_N \tag{19}$$

and therefore, from (18) and (19)

$$\mathcal{D}_n x^m = (\mathcal{D}_n x^{m-1})x + 2^{n-1}\left[x^{m-1}\right]_N e_N. \tag{20}$$

Now we prove by induction on m that $\mathcal{D}_n x^m$ vanishes on the quadratic cone $\mathcal{Q}_n$. For $m = 1$, $\mathcal{D}_n x = 0$ on the whole algebra. Let $m > 1$ and assume that $\mathcal{D}_n x^{m-1} = 0$ on every point of $\mathcal{Q}_n$. Since the power function maps $\mathcal{Q}_n$ in $\mathcal{Q}_n$, for every $x \in \mathcal{Q}_n$ we have $[x^{m-1}]_N = 0$. The equality (20) and the inductive hypothesis allow to conclude that $\mathcal{D}_n x^m = 0$ at $x \in \mathcal{Q}_n$.

Case (3): $n = 2$. Formula (20) is valid also for $n = 2$. Since the power function maps the set of reduced quaternions to itself, the inductive argument given above for case (2) shows that $\mathcal{D}_2 x^m$ vanishes at every point x of $\mathbb{R}_2$ with $x_{12} = 0$, i.e. at every reduced quaternion. □

Corollary 7. *Let $n \geq 3$. Let $p(x) = \sum_{j=0}^m x^j a_j$ be a polynomial in the complete Clifford variable $x = \sum_{|K|\leq n} x_K e_K$ with right Clifford coefficients. If n is odd, then p is in the kernel of $\mathcal{D}_n$. If n is even, then the restriction of $\mathcal{D}_n(p)$ to the quadratic cone $\mathcal{Q}_n$ vanishes.*

Polynomials $p(x) = \sum_{j=0}^m x^j a_j$ and convergent power series $\sum_k x^k a_k$ with right Clifford coefficients are examples of *slice regular* functions on the intersection of $\mathcal{Q}_n$ with a ball centered in the origin (cf. [6, 7] for this function theory). If $n \geq 3$, slice regularity generalizes the concept of *slice monogenic functions* introduced in [3]: if f is slice regular on a domain which intersects the real axis, then the restriction of f to the paravectors is a slice monogenic function and conversely. Since every slice monogenic function has a power expansions centered at real points, every slice monogenic or slice regular function on a domain Ω with $\Omega \cap \mathbb{R} \neq \emptyset$ satisfies the property stated in Corollary (7).

Remark 3. For $n = 1, 2$ the operators $\mathcal{D}_n$ are elliptic, since in this case

$$4\overline{\mathcal{D}}_n\mathcal{D}_n = 4\mathcal{D}_n\overline{\mathcal{D}}_n = \Delta_{\mathbb{R}^{2^n}}. \tag{21}$$

For $n = 3$ it holds

$$4\overline{\mathcal{D}}_3\mathcal{D}_3 = 4\mathcal{D}_3\overline{\mathcal{D}}_3 = \Delta_{\mathbb{R}^8} + \mathcal{L}_3, \tag{22}$$

where $\mathcal{L}_3 = -2\left(\partial_{x_0}\partial_{x_{123}} - \partial_{x_1}\partial_{x_{23}} + \partial_{x_2}\partial_{x_{13}} - \partial_{x_3}\partial_{x_{13}}\right)e_{123}$. For $n \geq 4$,

$$4\overline{\mathcal{D}}_n\mathcal{D}_n = \Delta_{\mathbb{R}^{2^n}} + \mathcal{L}_n \quad \text{and} \quad 4D_n\overline{\mathcal{D}}_n = \Delta_{\mathbb{R}^{2^n}} + \mathcal{L}'_n \tag{23}$$

where $\mathcal{L}_n = \sum_{H\neq K} t(e_H^*\tilde{e}_K)\partial_{x_H}\partial_{x_K}$ and $\mathcal{L}'_n = \sum_{H\neq K} t(\tilde{e}_H e_K^*)\partial_{x_H}\partial_{x_K}$ are different operators (the summations are made over multindices H, K without repetitions). In particular, for $n \geq 3$ the operators $\mathcal{D}_n$ are not elliptic. Note that the symbol of the differential operator $\mathcal{L}_3$ is, up to a multiplicative constant, the polynomial $x_0x_{123} - x_1x_{23} + x_2x_{13} - x_3x_{12}$ whose zero set is the *normal cone* of the Clifford algebra $\mathbb{R}_3$ (cf. [6] for its definition). A similar relation holds for the symbols of $\mathcal{L}_n$ and $\mathcal{L}'_n$ and the equations of the normal cone of $\mathbb{R}_n$ for $n > 3$.

3 THE CASE OF $\mathcal{D}_3$

3.1 Algebraic decomposition

Denote by $I = e_{123}$ the pseudoscalar of $\mathbb{R}_3$, and by $I_\pm = \frac{1}{2}(1 \pm I)$ the central elements with properties $I_\pm^2 = I_\pm,\ I_+I_- = 0,\ I_+ + I_- = 1$.

Let $X = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ be the paravector variable and $X' = x - X = x_{12}e_{12} + x_{13}e_{13} + x_{23}e_{23} + x_{123}e_{123}$. We can define two new (rotated) paravector variables $Y = y_0 + y_1e_1 + y_2e_2 + y_3e_3$ and $Z = z_0 + z_1e_1 + z_2e_2 + z_3e_3$ by setting

$$2Y = X + X'I, \quad 2Z = X - X'I,$$

from which we get the decomposition

$$x = X + X' = Y + Z + (Y - Z)I = 2YI_+ + 2ZI_-. \tag{24}$$

Since product by $I_\pm$ gives two orthogonal projections, for every positive integer m it holds

$$x^m = (2Y)^m I_+ + (2Z)^m I_-, \tag{25}$$

and therefore for every polynomial, power series or in general for a slice regular function f on a domain which intersect the real axis, we can write

$$f(x) = f(2Y)I_+ + f(2Z)I_-. \tag{26}$$

The operator $\mathcal{D}_3$ decomposes as $\mathcal{D}_3 = \frac{1}{2}(\partial_X - \partial_{X'})$, where $\partial_X = \partial_{x_0} + e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3}$ is the Weyl operator of $\mathbb{R}_3$ and $\partial_{X'} = e_{12}\partial_{x_{12}} + e_{13}\partial_{x_{13}} + e_{23}\partial_{x_{23}} + e_{123}\partial_{x_{123}}$. Denote by ∂_Y and ∂_Z the Weyl operators w.r.t. Y and Z respectively. Then

$$2\partial_X = \partial_Y + \partial_Z, \quad 2\partial_{X'} = (\partial_Y - \partial_Z)I, \tag{27}$$

and therefore in the variables Y, Z the operator $\mathcal{D}_3$ has the following form:

$$\mathcal{D}_3 = I_-\partial_Y + I_+\partial_Z = \partial_Y I_- + \partial_Z I_+. \tag{28}$$

This decomposition implies that a function f belongs to the kernel of $\mathcal{D}_3$ if and only if its projections $f_- := fI_-$ and $f_+ := fI_+$ belong to the kernels of the Weyl operators ∂_Y and ∂_Z respectively. In particular, every pair of arbitrary functions $g(Y)$, $h(Z)$ define a function $f(Y, Z) = I_-h(Z) + I_+g(Y)$ in the kernel of $\mathcal{D}_3$. This property shows again that $\mathcal{D}_3$ is not an elliptic operator, as can be seen also from formula (22) expressed in the variables Y, Z:

$$4\overline{\mathcal{D}}_3\mathcal{D}_3 = 4\mathcal{D}_3\overline{\mathcal{D}}_3 = \frac{1}{2}(\Delta_Y + \Delta_Z) - \frac{1}{2}(\Delta_Y - \Delta_Z)I = I_-\Delta_Y + I_+\Delta_Z \tag{29}$$

where Δ_Y is the Laplacian w.r.t. the variables y_0, y_1, y_2, y_3 and similarly for Δ_Z.

3.2 The space $\mathcal{F}(\Omega)$

In view of the non-ellipticity of $\mathcal{D}_3$, we consider a proper subspace of $\ker \mathcal{D}_3$. Let $\Delta_X = \partial_X \overline{\partial}_X$, $\Delta_{X'} = \partial_{X'} \overline{\partial}_{X'}$ and $\Delta = \partial_X \overline{\partial}_X + \partial_{X'} \overline{\partial}_{X'} = \Delta_{\mathbb{R}^8}$.

Definition 4. Let Ω be an open subset of $\mathbb{R}_3$. We define

$$\mathcal{F}(\Omega) := \{f \in C^1(\Omega) \,|\, \mathcal{D}_3 f = 0,\ \Delta_X \partial_X f = 0 \text{ on } \Omega\}.$$

Proposition 8. *Let $\Omega \subseteq \mathbb{R}_3$ be open. The space $\mathcal{F}(\Omega)$ can be expressed in the variables Y, Z as*

$$\mathcal{F}(\Omega) = \{f \in C^1(\Omega) \,|\, \mathcal{D}_3 f = 0,\ \Delta_Y \partial_Y f = \Delta_Z \partial_Z f = 0 \text{ on } \Omega\}.$$

Every $f \in \mathcal{F}(\Omega)$ is biharmonic on Ω (i.e. $\Delta^2 f = 0$) and also biharmonic w.r.t. the variables Y and Z separately. In particular, it is real analytic on Ω. Moreover, $f = f_- + f_+ \in \mathcal{F}(\Omega)$ if and only if its projections f_- and f_+ satisfy

$$\partial_Y f_- = \Delta_Z \partial_Z f_- = 0, \quad \partial_Z f_+ = \Delta_Y \partial_Y f_+ = 0. \tag{30}$$

Proof. If $\mathcal{D}_3 f = 0$, then $\partial_X f = \partial_{X'} f$. Therefore $\Delta f = (\partial_X \overline{\partial}_X + \partial_{X'} \overline{\partial}_{X'}) f = 2\Delta_X f$. Moreover, from (27) it follows that $\partial_Z f = (\partial_X - I\partial_{X'}) f = (\partial_X - I\partial_X) f = 2I_- \partial_X f$. Then $\Delta_Z f = \partial_Z \overline{\partial}_Z f = 4I_- \partial_X \overline{\partial}_X f = 4\Delta_X f_- = 2\Delta f_-$ and therefore $\Delta_Z \partial_Z f = 8\Delta_X \partial_X f_-$. A similar computation gives $\Delta_Y \partial_Y f = 8\Delta_X \partial_X f_+$. Then $\Delta_X \partial_X f = 0$ if and only if $\Delta_Z \partial_Z f = \Delta_Y \partial_Y f = 0$.

If $f \in \mathcal{F}(\Omega)$, then $0 = \overline{\partial}_Z \partial_Z \Delta_Z f = \Delta_Z^2 f$ and $0 = \overline{\partial}_Y \partial_Y \Delta_Y f = \Delta_Y^2 f$. From these equalities we get $4\Delta^2 f_- = \Delta_Z^2 f = 0$, $4\Delta^2 f_+ = \Delta_Y^2 f = 0$ and then $\Delta^2 f = 0$.

The last statement is immediate from (28). □

Remark 4. The preceding Proposition tells that every function in the space $\mathcal{F}(\Omega)$ is (separately) *holomorphic Cliffordian* (cf. [8]) in the paravector variables X, Y and Z.

Corollary 9. *Every polynomial $p(x) = \sum_{j=0}^m x^j a_j$ in the Clifford variable $x = \sum_{K \in \mathcal{P}(3)} x_K e_K$ belongs to $\mathcal{F}(\mathbb{R}_3)$. The same holds for every slice regular function on a domain in $\mathcal{Q}_3$ intersecting the real axis. If $f(X)$ is a function depending only on the paravector variable X of $\mathbb{R}_3$, then $f \in \mathcal{F}$ if and only if it is monogenic, i.e. $\partial_X f = 0$.*

Proof. From (25) every power of x can be expressed by means of powers of Y and Z. Since every power of a paravector variable X is holomorphic Cliffordian [8], i.e. $\Delta_X \partial_X f = 0$, the first two statements follow from Theorem (5) and Corollary (7). The last statement is an immediate consequence of Corollary (2). □

Let B denote the eight-dimensional unit ball in $\mathbb{R}_3$. Let $T \simeq S^3 \times S^3$ be the subset of the unit sphere ∂B defined by $T := \{|Y| = |Z| = 1/2\}$ and $P := \{|Y| < 1/2\} \cap \{|Z| < 1/2\}$. Since $|x|^2 = 2|Y|^2 + 2|Z|^2$, $P \subset B$ and $T \subset \partial P$.

Note that T is contained in the normal cone $\mathcal{N}_3$, which has equation $|Y| = |Z|$ in the variables Y, Z.

Proposition 10 (Integral Representation Formula)**.** *There is an integral representation formula for functions $f \in \mathcal{F}(P) \cap C^2(\overline{P})$ with T as domain of integration. The values of f on P are determined by the values on T of f, $\partial_X f$ and the second derivatives $\frac{\partial}{\partial x_K}(\partial_X f)$.*

Sketch of the proof. Consider the component $f_- \in \mathcal{F}(P)$. Since $\partial_Y f_- = 0$, we can apply the representation formula for the Weyl operator ∂_Y (cf. [1]) and reconstruct f_- on the set $\{|Y| < 1/2, |Z| = 1/2\}$. Since $\partial_Y \partial_Z f_- = 0$, we can do the same for $\partial_Z f_-$. Since $\Delta_Z \partial_Z f_- = 0$, we can now apply the integral representation formula for holomorphic Cliffordian functions (see [8]) w.r.t. the paravector variable Z and obtain the values of f_- on P. A similar reasoning for f_+ gives the result. □

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