The linking number magic

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We study the finite element construction of loop fields and vector potentials in a general Lipschitz polyhedral domain Ω. By loop fields we mean irrotational vector fields that cannot be expressed in Ω as the gradient of any single-valued scalar potential. For both problems we provide an explicit formula of the solution based on the computation of certain linking numbers. We use a homological approach: the key point is the knowledge of a set of 1-cycles $\{\gamma_j\}_{j=1}^g$ that are representatives of a basis of the first homology group $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$ of $\overline{\Omega}$.

For the construction of the loop fields we need also a basis of $\mathcal{H}_1(\mathbb{R}^3 \setminus \Omega, \mathbb{Z})$, $\{\gamma'_j\}_{j=1}^g$. Given a spanning tree of the vertices and edges of the mesh, we construct a maximal set of independent loop fields in terms of the linking number between some 1-cycles of the mesh, constructed using the spanning tree, and the 1-cycles γ'_j.

On the other hand we construct a finite element vector potential in terms of the 1-cycles $\{\gamma_j\}_{j=1}^g$ and homological Seifert surfaces. Given a 1-boundary σ of the mesh, a homological Seifert surface of γ is a 2-chain S of the mesh such that $\partial S = \sigma$. The linking number turns to be again a key tool in the identification of these surfaces. Given a spanning tree of the dual mesh, the coefficients of the 2-chain S are the linking number between certain 1-cycles of the dual mesh related to the spanning tree and σ.

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