The boundary of (multi-)vector fields

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The intuitive notion of boundary is basically topological. Also the concept of boundary of a chain, while more elaborate, has a clear intuitive meaning, consistent with the basic one. Attaching the same term to vector fields may seem far-fetched. The primary aim of this talk is to strive to convince you that the boundary of a vector field (and, more generally, of a k-vector field) makes good sense. To this end, I gather and use three scattered tools from exterior calculus: i) the definition of contravariant exterior derivative, mentioned in passing by Abraham, Marsden and Ratiu in their extended introduction to nonlinear analysis [1]; ii) the notion of k-measure, introduced by Fichera in a little-known paper [2] (written in Italian and never translated into English, as far as I know); iii) the intrinsic definition of the exterior derivative in terms of Lie derivatives, due to Palais [3].

I define the boundary operator on multi-vector fields in such a way that the exterior derivative acting on multi-covector fields is dual to it: the duality between the exterior derivative of a $(k-1)$-form and a (locally summable) k-vector field equals the duality between the $(k-1)$-form itself and the boundary of the k-vector field. Basically, Fichera's k-measures are k-vector-valued measures. In particular, k-measures absolutely continuous with respect to the Lebesgue measure may be identified with locally summable k-vector fields. The boundary of a k-measure is not necessarily a $(k-1)$-measure, being possibly a distribution of higher degree. I introduce a selected subset of the space of k-measures, the set of k-bricks, and its span, the subspace of k-assemblages. The support of a k-brick is a k-cell and its boundary is a $(k-1)$-assemblage, supported by the (topological) boundary of the k-cell. This results reconciles the different notions of boundary and justifies my terminology. What's more, it allows the same format to cover both differential and discrete exterior calculus, k-chains and k-cochains being paralleled by k-vector fields and k-forms, respectively. I will elaborate on the potential benefits of this unification to computational electromagnetics.

References

[1] R. Abraham, J.E. Marsden, T. Ratiu, Manifolds, Tensor Analysis, and Applications (Second Edition), Applied Mathematical Sciences 75, Springer, New York (1988).

[2] G. Fichera, Spazi lineari di k-misure e di forme differenziali, Proceedings of the International Symposium on Linear Spaces held at the Hebrew University of Jerusalem, July 5-12, 1960 (1961), 175-226.

[3] R.S. Palais, A definition of the exterior derivative in terms of Lie derivatives, Proceedings of the American Mathematical Society 5 (1954), 902-908.