Space-Time Discretization of Maxwell's Equations in the Setting of Geometric Algebra

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The Geometric Algebra (GA) language enables to write Maxwell Equations (ME) in a very elegant form, similar to Differential Forms approach. Moreover, the space-time integral form, i.e., all differential operators, including time differentiation, are replaced by integral ones, of ME can be easily obtained and investigated. This form of ME consists of two equations, which in principle can be applied to a set of meshes (primal and dual) in the space-time domain.

We show that all discrete values of electric and magnetic field strength and flux should live on two-dimensional space-time objects ('facets') of the primal and dual mesh. This choice is in agreement with Tonti's Cell Method approach but it is unlike the classical 3-dimensional space approach where the degrees of freedom are obtained by spatial integration over edges (e.g. for the electric field strength E) and facets (e.g. magnetic flux density B).

Due to the fact that electric and magnetic fields are treated on equal footing, we can also express the constitutive equations (CE) without Ohm's law as one multivector equation. It is discretized to relate the discrete values of the fields on the dual and primal meshes. Imposing an orthogonality condition ensures one-to-one relations, i.e., a discrete material relation involves only one quantity from each mesh. This turns the corresponding material matrices to be diagonal in the case of scalar material parameters on Cartesian grids where our space-time approach can be shown to be equivalent to (generalized) Yee's Schemes and the Finite Integration Technique (classical Hodge duality). However, it is not longer the case when we allow for anisotropy. Doing so, we need to change orthogonality to more general condition.

Finally we discuss Ohm's law. One of its unique attributes in space-time which sets it apart from other CE is the fact that it relates the electric field, whose discrete equivalent lives on 2D facets, with the current density, whose discrete equivalent lives on 3D facets. Furthermore, the consistency of Ohm's law with special relativity is not manifest.