Green's double forms for the Hodge-Laplacian on Riemannian manifolds with constant curvature: a case-study in two-dimensions

Stefan Kurz
Tampere University of Technology
Department of Electrical Engineering, Electromagnetics
33101 Tampere, Finland
E-mail: stefan.kurz@tut.fi

Abstract

In [1], boundary integral equations for Maxwell-type problems have been discussed in terms of differential forms. Such problems are governed by the equation $(\delta \mathrm{d} - k^2)\boldsymbol{\omega} = 0, \quad k \in \mathbb{C}$, where d denotes the exterior derivative, δ the co-derivative, and ω is a differential form of degree p. This problem class generalizes **curl curl**- and div **grad**-types of problems in three dimensions. In the differential forms framework, kernels of integral transformations are described by double forms. A double p-form is defined for $p > 0$ by its action on a pair of p-tupels of tangent vectors anchored in observation point X and source point Y, respectively. The relevant Green's double form is given as fundamental solution of the Helmholtz equation, $(\Delta - \overline{k}^2)G = \delta(X,Y)I$. Herein, Δ is the Hodge-Laplacian, G the Green's double form, δ the Dirac delta distribution, and I the identity double form. In contrast to the classical vector analysis formulation in Euclidean space, the differential forms-based formulation and analysis remains valid in curved spaces as well. Some care has to be taken regarding solvability, though.

In flat space, the Green's double form of bi-degree p>0 can be easily constructed from scalar Green's function (p=0) and the identity double form of bi-degree p. This construction leverages the fact that in flat space there is a path-independent notion of parallel transport. In general, the Green's double forms need to be derived for each bi-degree p separately, though.

We consider Riemannian manifolds with constant curvature. It turns out that Green's double forms can be reduced to certain solutions of the hypergeometric differential equation [2].

The simplest case n=2, p=0 with constant positive curvature has been treated in [3]. The Green's function can be interpreted as electrostatic potential of a point charge on the sphere, with an opposite point charge in the antipodal point. Hodge duality yields the case n=2, p=2. However, an explicit expression of Green's double form for n=2, p=1 seems to be lacking from literature so far.

In this contribution, a closed-form expression for this Green's double form is derived, hence enabling integral operators for magnetostatics on the sphere. To that end, concepts like geodesic distance, parallel propagator and Fermi normal coordinates [4] are invoked. Also, the feasibility of the differential forms framework in a non-Euclidean space is highlighted.

References

[1] S. Kurz and B. Auchmann. Differential Forms and Boundary Integral Equations for Maxwell–Type Problems. In: U. Langer, M. Schanz, O. Steinbach, and W. L. Wendland (eds.). Fast Boundary Element Methods in Engineering and Industrial Applications. Springer Lecture Notes in Applied and Computational Mechanics, Volume 63/2012: 1-62.

[2] A. Enciso and N. Kamran. Green's Function for the Hodge Laplacian on Some Classes of Riemannian and Lorentzian Symmetric Spaces. Commun. Math. Phys., 290: 105-127, 2009.

[3] S. Gemmrich, N. Nigam and O. Steinbach. Boundary Integral Equations for the Laplace-Beltrami Operator. In: H. Munthe-Kaas and B. Owren (eds.) Mathematics and Computation, a Contemporary View, Springer Berlin Heidelberg, Volume 3/2008: 21-37.

[4] F. Manasse and C. Misner, Fermi normal coordinates and some basic concepts in differential geometry, J. Math. Phys., 4 (1963), pp. 735–745.

Abstract submitted 12.02.2013. The presentation was also held at the 11. Workshop Fast Boundary Element Methods in Industrial Applications, 26.09.-29.09.2013, Söllerhaus, see http://www.numerik.math.tu-graz.ac.at/tagungen