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Method of moments: $\gamma(0) = \hat{\gamma}(0), \quad \rho(1) = \hat{\rho}(1)$. Hence

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It may have no solutions (if $|\hat{\rho}(1)| > 1/2$) or 2 solutions (if $|\hat{\rho}(1)| > 1/2$) of whoch we may choose that yielding an **invertible** process.
Moreover, it can be shown that it is **not efficient** (there are other simple estimators with asymptotically lower variance).

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$\hat{v}_m$ an estimator of σ^2.

Fitted innovations algorithm. Intuitive explanation

$$X_{m+1} = Z_{m+1} + \vartheta_1 Z_m + \ldots + \vartheta_q Z_{m-q+1}.$$

Innovations algorithm on the true ACVF gives

$$\hat{X}_{m+1} = \vartheta_{m,1}(X_m - \hat{X}_m) + \ldots + \vartheta_{m,m}(X_1 - \hat{X}_1).$$

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$$X_{m+1} = (X_{m+1} - \hat{X}_{m+1}) + \hat{X}_{m+1}$$
$$= (X_{m+1} - \hat{X}_{m+1}) + \vartheta_{m,1}(X_m - \hat{X}_m) + \vartheta_{m,m}(X_1 - \hat{X}_1)$$

we expect $\vartheta_{m,j} \approx \vartheta_j$ for $j = 1 \ldots q$, $\quad \vartheta_{m,j} \approx 0$ for $j > q$.

Convergence of fitted innovations estimator

This theorem (Brockwell-Davis, 1988) is stated for ARMA(p,q) processes

Theorem

Let $\{X_t\}$ *be the* **causal, invertible** *ARMA process specified by* $\varphi(B)X = \vartheta(B)Z$ *with* $\{Z_t|\} \sim IID(0, \sigma^2)$ *and* $\mathbb{E}(|Z_t|^4 < \infty$, *and let* $\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\vartheta(z)}{\varphi(z)}$. *Then for any sequence* m_n *such that* $m_n < n$, $\lim_{n\to\infty} m_n = +\infty$ *and* $\lim_{n\to\infty} n^{-1/3} m_n = 0$ *and for any* $k > 0$

$$n^{1/2}(\vartheta_{m_n,1} - \psi_1, \ldots \vartheta_{m_n,k} - \psi_k) \Longrightarrow N(0, A) \quad \text{as } n \to \infty$$

where $A_{ij} = \sum_{r=1}^{\min(i,j)} \psi_{i-r}\psi_{j-r}$.

Moreover $\hat{v}_m \to \sigma^2$ *in probability.*

Remarks on convergence of fitted innovations

- $(\hat{\vartheta}_{q,1}, \ldots, \hat{\vartheta}_{q,q})$ is not a consistent estimator of $(\vartheta_1, \ldots, \vartheta_q)$. *It is not enough that $n \to \infty$, need also $m \to \infty$ (with appropriate speed).*

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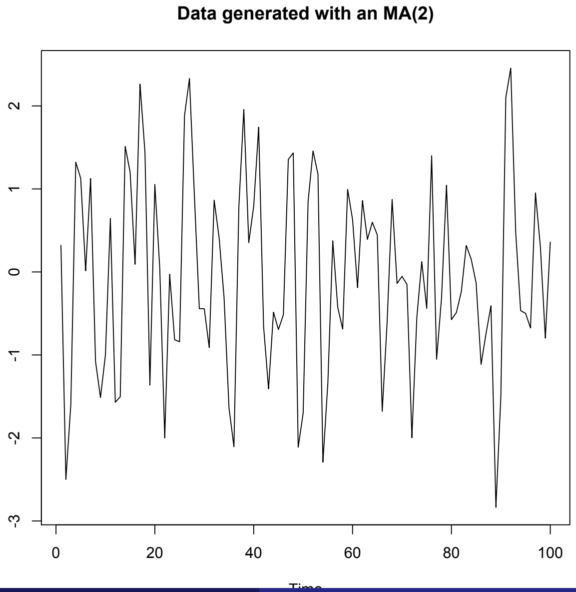
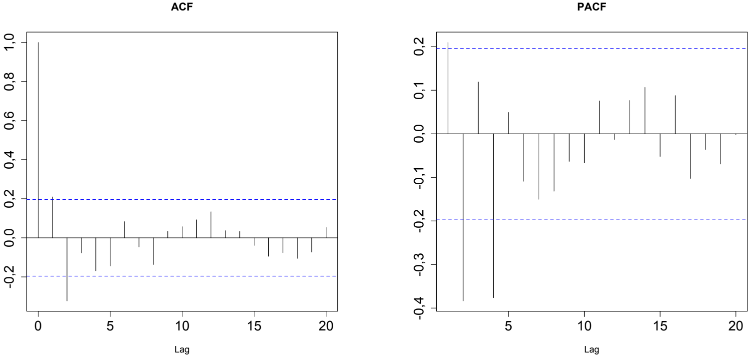
- in practice one has finite n and has to choose a finite m. Increase m until $(\hat{\vartheta}_{m,1}, \ldots, \hat{\vartheta}_{m,q})$ stabilize.
- asymptotic confidence intervals can be found from the elements of the matrix A.

An example on simulated data



ACF and PACF of simulated data



An MA(2) process is suggested by these plots

Using the fitted innovations algorithm

m	$\vartheta_{m,1}$	$\vartheta_{m,2}$	v_m
2	0.290	-0.323	1.028
3	0.336	-0.321	1.015
4	0.380	-0.448	0.977
5	0.399	-0.436	0.978
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10	0.355	-0.508	0.971
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Confidence intervals:

$$A_{11} = 1 \quad \hat{A}_{22} = 1 + 0.36^2 \approx 1.1296$$

$$\vartheta_1 = 0.36 \pm 1.96\sqrt{\frac{1}{n}} = 0.36 \pm 0.20$$

$$\vartheta_2 = -0.507 \pm 1.96\sqrt{\frac{1.1296}{n}} = -0.51 \pm 0.21$$

'True' values were
$\vartheta_1 = 0.5$, $\vartheta_2 = -0.4$.

Using innovations for an ARMA(p,q)

Fitted innovations yield estimates $\hat{\vartheta}_{m,1}$, $\hat{\vartheta}_{m,2}, \dots$ for ψ_1, $\psi_2, \dots$.
As in an ARMA(p,q)

$$\psi_j = \vartheta_j + \sum_{i=1}^{\min(j,p)} \varphi_i \psi_{j-i} \quad j = 1, 2, \dots$$

we set

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The equations for $j = q+1, \dots, q+p$ do not involve $\hat\vartheta$ and can be solved for $\hat\varphi_1, \dots, \hat\varphi_p$.
Then equations for $j = 1, , \dots, q$ yield $\hat\vartheta_j$ directly.

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Then equations for $j = 1, , \dots, q$ yield $\hat{\vartheta}_j$ directly.

Note that $\hat{\varphi}_1, \dots, \hat{\varphi}_p$ do not necessarily yield a **causal** process.

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Example of AR(1): $X_t = \mu + \varphi(X_{t-1} - \mu) + Z_t$, $|\varphi| < 1$.
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Hence likelihood of data $(x_1, x_2, \ldots, x_n)$ can be obtained as

$$f(x_1)f(x_2|x_1)\cdots f(x_n|x_{n-1})$$

where $f(\cdot)$ is the (unconditional) density of X_t,
while $f(\cdot|x)$ is the density of X_t conditional on $X_{t-1} = x$.

Computing likelihood for AR(1)

$$f(x) = (2\pi\frac{\sigma^2}{1-\varphi^2})^{-1/2}\exp\{-\frac{(x-\mu)^2}{2\sigma^2/(1-\varphi^2)}\}$$
$$= \left(\frac{1-\varphi^2}{2\pi\sigma^2}\right)^{1/2}\exp\{-\frac{(1-\varphi^2)(x-\mu)^2}{2\sigma^2}\}$$
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Parameters of the model are μ, φ, σ^2:

$$L(\mu,\varphi,\sigma^2) = (1-\varphi^2)^{1/2}(2\pi\sigma^2)^{-n/2} \exp\{-\frac{(1-\varphi^2)(x_1-\mu)^2}{2\sigma^2}\}$$

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Maximum likelihood estimation: example of AR(1)

Assume $X_t = \mu + \varphi(X_{t-1} - \mu) + Z_t$ with $\{Z_t\} \sim IID\ N(0, \sigma^2)$. Then the likelihood given data $(x_1, x_2, \ldots, x_n)$ is

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where $\hat{\mu}$ and $\hat{\varphi}$ are MLE. Now substitute $\hat{\sigma^2}$ and maximize w.r. to (μ, φ).

MLE and least square estimation for AR(1)

Taking logarithm of L, substituting $\hat{\sigma^2}$ and ignoring additive constants yields $-\frac{n}{2}\log(n^{-1}S(\mu,\varphi)) + \frac{1}{2}\log(1-\varphi^2)$.

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or with a further simplification by *conditional* least squares, i.e. minimize

$$S_c(\mu,\varphi) = \sum_{t=2}^{n}(x_t-\mu-\varphi(x_{t-1}-\mu))^2.$$

Conditional least squares

$$S_c(\mu, \varphi) = \sum_{t=2}^{n} (x_t - \mu - \varphi(x_{t-1} - \mu))^2 = \sum_{t=2}^{n} (x_t - (\alpha + \varphi x_{t-1}))^2$$

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$$i.e.\ \hat{\mu}_c = \frac{\bar{x}_{(2)} - \hat{\varphi}_c \bar{x}_{(1)}}{1 - \hat{\varphi}_c}$$

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$$i.e.\ \hat{\mu}_c = \frac{\bar{x}_{(2)} - \hat{\varphi}_c \bar{x}_{(1)}}{1-\hat{\varphi}_c} \approx \bar{x}$$

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Estimates very similar to Yule-Walker. $\hat\mu \approx \bar x$ often suggests to assume $Y_t = X_t - \bar X$ has mean 0 in more complex models.

Likelihood function

Assume $\{X_t\}$ is a 0-mean ARMA(p,q) process with Gaussian distribution,

$$\mathbb{X}_n = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ the random vector} \quad \text{and} \quad \mathbf{x}_n = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ the observed data.}$$

The likelihood can be written in terms of Γ_n, the covariance matrix of $\mathbb{X}_n$:

$$L(\Gamma_n) = (2\pi)^{-n/2} \left|\Gamma_n\right|^{-1/2} \exp\left\{-\frac{1}{2}\mathbf{x}_n^t \Gamma_n^{-1} \mathbf{x}_n\right\}.$$

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$|\Gamma_n|$ and $\mathbf{x}_n^t \Gamma_n^{-1} \mathbf{x}_n$ can be computed with the innovations algorithm.

Recursive computation of the likelihood

We showed $\mathbb{X}_n = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = C_n(\mathbb{X}_n - \hat{\mathbb{X}}_n)$, where $C_n = \begin{pmatrix} 1 & 0 & 0 \\ \bullet & \ddots & \vdots \\ \bullet & \bullet & 1 \end{pmatrix}$ and

$$\mathbb{V}(\mathbb{X}_n - \hat{\mathbb{X}}_n) = D_n = \text{diag}(v_0 \cdots v_{n-1}) \text{ with } v_j = \mathbb{E}(X_{j+1} - \hat{X}_{j+1})^2.$$

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Finally $|\Gamma_n| = |C_n|^2 |D_n| = \prod_{i=0}^{n-1} v_i$, as C_n is triangular with all 1's on the diagonal.

Likelihood for an ARMA process: final expressions

Let $(\phi, \vartheta, \sigma^2)$ the parameters of an ARMA model. We found

$$L(\phi, \vartheta, \sigma^2) = (2\pi)^{-n/2}(v_0 \dots v_{n-1})^{-1/2} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(x_j - \hat{x}_j)^2}{v_{j-1}} \right\}$$

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Applying the innovations to $W_t = \begin{cases} \sigma^{-1} X_t & t \leq m = \max\{p, q\} \\ \sigma^{-1}(\Phi(B)X)_t & t > m \end{cases}$

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  - MLE estimate is searched among those causal. Hence the initial estimate has to be causal (*Yule-Walker and Burg ensure this*).
- $r_0, \ldots, r_{n-1}$ depend only on parameters (ϕ, ϑ) and not on observed data; the same is true for $\vartheta_{n,j}$ of innovations algorithm. Data enter likelihood only through the terms $(x_j - \hat{x}_j)^2$ in $S(\phi, \vartheta)$.

Asymptotic distribution of MLE

Theorem

Let $\{X_t\}$ *be a causal and invertible ARMA(p,q) process satisfying*

$$\Phi(B)X = \Theta(B)Z, \qquad \{Z_t\} \sim IID(0, \sigma^2).$$

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$$W = \sigma^2 \begin{pmatrix} \mathbb{E}(\mathbb{U}_t \mathbb{U}_t^t) & \mathbb{E}(\mathbb{U}_t \mathbb{V}_t^t) \\ \mathbb{E}(\mathbb{V}_t \mathbb{U}_t^t) & \mathbb{E}(\mathbb{V}_t \mathbb{V}_t^t) \end{pmatrix}^{-1} \text{with } \mathbb{U}_t = \begin{pmatrix} U_t \\ \vdots \\ U_{t-p+1} \end{pmatrix} \ \mathbb{V}_t = \begin{pmatrix} V_t \\ \vdots \\ V_{t-q+1} \end{pmatrix}$$

$$(\Phi(B)U)_t = Z_t \qquad (\Theta(B)V)_t = Z_t.$$