BIRATIONAL MODELS OF MORI FIBRE SPACES, PLIABILITY AND COX RINGS

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Abstract. We study birational transformations of certain fibtations of degree 4 del Pezzo surfaces over $\mathbb{P}^1$, into other Mori fibre spaces, using Cox rings. We show that these Mori fibre spaces have a (relatively) big pliability although they are not rational. Our methods can be applied to study birational geometry of Mori dream spaces with low rank Cox ring.

Contents

1. Introduction

Mori fibre spaces are outcomes of the minimal model programme for varieties with negative Kodaira dimension. Formally, a normal variety X with at worst $\mathbb{Q}$-factorial terminal singularities is called a Mori fibre space if there exists a morphism $\varphi\colon X \to Z$ to a normal variety Z, of strictly smaller dimension than X, such that $-K_X$ is φ-ample and the relative Picard number is equal to one, that is $\rho(X/Z) = 1$. It is crucial to investigate how many different Mori fibre spaces fall in the same birational class, and study their properties. Naturally, this problem is considered modulo *square birational.*

Definition 1.1. Let $X \to Z$ and $X' \to Z'$ be Mori fibre spaces. A birational map $f\colon X \dashrightarrow X'$ is *square* if it fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d] & X' \arrow[d] \\ S \arrow[r, dashed, "g"] & S' \end{tikzcd}$$

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where g is a birational map and, in addition, the induced birational map of generic fibres $f_L\colon X_L \dashrightarrow X'_L$ is biregular. In this case we say that X/Z and X'/Z' are *square birational.*

Definition 1.2. The *pliability* of a Mori fibre space $X \to S$ is the set

$$\mathcal{P}(X/S) = \{\text{Mfs } Y \to T \mid X \text{ is birational to } Y\}/\text{square equivalence}.$$

We sometimes abuse the term pliability to mean the cardinality of this set, when it is finite.

For many Mori fibre spaces this number is known to be very small or the variety is known to be rational, and hence have a very big pliability. For instance smooth fibrations of del Pezzo surfaces over $\mathbb{P}^1$ with degree of the general fibre five or above are known to be rational. On the opposite side general Fano hypersurfaces in a weighted projective space are birationally rigid [10], hence have pliability one. Many other similar results have been obtained for 3-fold Mori fibre spaces, in both directions. An interesting model with pliability exactly two is constructed as a quartic in $\mathbb{P}^4$ with a cA_2 singular point [9].

In this article we consider a model of del Pezzo fibration of degree 4 over $\mathbb{P}^1$ and show that its pliability is at least 3. Then using some known results we prove it is not rational. We recall that the degree of a del Pezzo surface is the self-intersection number of its canonical class. Birational geometry of del Pezzo fibrations, considered as Mori fibre spaces, plays an important role in the theory of classification of algebraic varieties in dimension 3. While degree 5 or above imply rationality, we expect more rigidity as the degree becomes lower. There are many results of this type on birational rigidity and nonrigidity for degree 1, 2 and 3 fibrations, see for example [1, 3, 5, 15, 19]. However, for degree 4 fibrations the only results concern the rationality problem [2, 21]. It is also shown in [2] that these varieties are birational to conic bundles over the base, which is not interesting in terms of minimal model theory. Our result also sheds light on the study of birational behaviour of the nonrational models.

The construction of the links between various models, by means of Sarkisov programme, are obtained via Cox embeddings. In [4] it was shown how to run type III or IV Sarkisov programmes on a rank 2 Cox rings. This method has been applied to many classes of Mori fibre spaces with Picard number 2, see [1,3]. Some of the models we construct are obtained in this fashion and for others we introduce, more general, methods of working with higher rank Cox rings and also explain how the maps between these Cox rings look like. In other words, these constructions provide explicit Sarkisov links of type I or II in the language of Cox data. The following is our main theorem.

Theorem 1.3. *Let $\mathcal{F}$ be the 5-fold $\mathcal{F} = \operatorname{Proj}_{\mathbb{P}^1} \mathcal{E}$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$. Let M be the class of the tautological bundle on F and L the class of a fibre (over $\mathbb{P}^1$). Then there are Q_1 and Q_2, two hypersurfaces in $\mathcal{F}$, with $Q_1 \in |-3L + 2M|$ and $Q_2 \in |-2L+2M|$ such that for $X = Q_1 \cap Q_2$, the complete intersection of these two hypersurfaces, we have*

(1) X is smooth with $\operatorname{Pic}(X) \cong \mathbb{Z}^2$ and $X \to \mathbb{P}^1$ is a Mori fibre space with generic fibre del Pezzo surface of degree 4,
(2) X is birational, but not square birational, to at least two other Mori fibre spaces,
(3) X is not rational.

Conditions on the generality of Q_1 and Q_2 are specified in Section 4.

The structure of the article is as follows. In Sections 2 we study well-fomedness of the homogeneous coordinate ring of toric varieties. In Section 3 we introduce explicit methods of working with blow ups of low rank Cox rings. Sectoions 2 and 3 can be read independently of this article and the results are quite general and do not restrict only to the cases studied in this article. Among applications of these methods are the description of the starting point of type I and II Sarkisov links, that we apply. Section 2 explains how some of the blow up varieties can be modified to simpler ones, isomorphically. This generalizes the notion of well-formedness of weighted projective spaces [17] to that of Cox rings. Equivalently, it is an explicit method of finding Cox ring of coarse moduli of toric Deligne-Mumford stacks [13]. In Section 4 we conclude our results and explicitly describe the links between varieties under study. Tools provided in earlier sections will be used frequently in the proofs.

2. Well-formedness and stacky models

Weighted projective spaces have been studied extensively, and the well-formedness property, as in citedol and [17] plays an important role in the basic theory.

A weighted projective space, denoted by $\mathbb{P}(a_0,\dots,a_n)$, for positive integers $a_0,\dots,a_n$, is defined by the geometric quotient of $\mathbb{C}^{n+1}-\{0\}$ when acted on by $\mathbb{C}^*$ via

$$\lambda.(x_0,\dots,x_n)\mapsto(\lambda^{a_0}x_0,\dots,\lambda^{a_n}x_n),\quad \text{for } \lambda\in\mathbb{C}^*.$$

In other words, $\mathbb{P}(a_0,\dots,a_n)=\operatorname{Proj}\mathbb{C}[x_0,\dots,x_n]$, where $\mathbb{C}[x_0,\dots,x_n]$ is $\mathbb{Z}$-graded with $\deg(x_i)=a_i$.

The weighted projective space $\mathbb{P}(a_0,\dots,a_n)$ is *well formed* if $\gcd(a_0,\dots,\hat{a_i},\dots,a_n)=1$ for all $0\le i\le n$. Constructing the well-formed model of a given quotient $\mathbb{P}(a_0,\dots,a_n)$ is done in two steps (see [17]):

(1) Removing generic stabilisers: Find $a=\gcd(a_0,\dots,a_n)$, then replace $\mathbb{P}(a_0,\dots,a_n)$ by $\mathbb{P}(b_0,\dots,b_n)$, where $b_i=\frac{a_i}{a}$.

(2) Removing quasi-reflections: For each b_i find $\mathbf{b_i}=\gcd(b_0,\dots,\hat{b_i},\dots,b_n)$ and replace $\mathbb{P}(b_0,\dots,b_n)$ by

$$\mathbb{P}(\frac{b_0}{\mathbf{b_i}},\dots,\frac{b_{i-1}}{\mathbf{b_i}},b_i,\frac{b_{i+1}}{\mathbf{b_i}},\dots,\frac{b_n}{\mathbf{b_i}}).$$

Remark 1. In the setting of [13], the variety obtained at Step 1 above is the toric orbifold associated to the toric Deligne-Mumford stack $\mathbb{P}(a_0,\dots,a_n)$ and the variety produced at the end (the well-formed model) is the corresponding coarse moduli, see [13] Example 7.27.

Note that for a given set of positive integers defining the weights, and positive integers α and β the following holds ([17] §5):

$$\mathbb{P}(a_0,\dots,a_n)\cong\mathbb{P}(\alpha a_0,\dots,\alpha a_n)\cong\mathbb{P}(a_0,\beta a_1,\dots,\beta a_n)$$

and this is exactly why one is permitted to do the process above and obtain isomorphic quotients. Our aim in this section is to obtain similar construction for projective toric Deligne-Mumford stacks. These (non-well formed) Cox rings arise naturally in our description of blow ups of Cox rings in the following sections.

Let T be a toric stack of dimension d determined by the fan Δ in $N\cong\mathbb{Z}^d$ and denote the set of 1-dimensional cones in Δ by $\Delta(1)$. Assume $M=\operatorname{Hom}(N,\mathbb{Z})$ is the dual lattice and $|\Delta(1)|=n$. The following exact sequence ([14], §3.4.) reads off the nature of divisors on T

$$0\longrightarrow M\longrightarrow\mathbb{Z}^n\xrightarrow{A}\operatorname{Cl}(T)\longrightarrow 0\quad. \tag{1}$$

The stack T can be realized as the quotient $(\mathbb{C}^n-V(I))/\!\!/G$, where I, the *irrelevant ideal*, is the ideal of the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$ obtained from the combinatorial structure of the fan Δ, and $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(T), \mathbb{C}^*)$. In particular, G is reductive and we have

$$G = (\mathbb{C}^*)^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}, \quad \text{for some } r, k, a_i \in \mathbb{N} \text{ and } r = n - d \quad .$$

Without loss of generality we can assume the torsion part, i.e. $\bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}$, induces no generic stabilisers nor quasi-reflections. Otherwise removing the corresponding part should be easy and produces isomorphic quotients as desired.

The aim is to identify inappropriate components of the non-torsion part that cause such behaviour and *remove* them. Let us assume, for now, for simplicity in writing, that $\mathrm{Cl}(T)$ is torsion free; in other words $k = 0$. Hence $\mathrm{Cl}(T) \cong \mathbb{Z}^r$ and it follows that $A \in \mathcal{M}_{r\times n}(\mathbb{Z})$. By applying the functor $\mathrm{Hom}(-, \mathbb{C}^*)$ to the short exact sequence (1) one obtains

$$1 \longrightarrow G \xrightarrow{A^*} (\mathbb{C}^*)^n \longrightarrow \mathbb{T} \longrightarrow 1 \quad , \tag{2}$$

where $\mathbb{T}$ is the torus acting on T. The action of G on $\mathbb{C}^n$ is the extension of the action on $(\mathbb{C}^*)^n$ above and is identified by the matrix $A = (a_{ij})$ above in the following way

$$(\lambda_1, \ldots, \lambda_r) \cdot x_j \mapsto \prod_{i=1}^{r} \lambda_i^{a_{ij}} x_j \quad .$$

Notation 2.1. We use $T = T(I, A)$ to denote the quotient of $(\mathbb{C}^n - Z)/\!/G$, where I is the irrelevant ideal, $Z = V(I) \subset \mathbb{C}^n$, $G = (\mathbb{C}^*)^r$ and the action is identified by the matrix $A \in \mathcal{M}_{r\times n}(\mathbb{Z})$ as before.

Definition 2.2. Let $T = T(I, A)$ be a toric stack. We define the rank of T to be $r = \mathrm{rank}\, A$.

Lemma 2.3. *Let $T = T(I, A)$ and $B = gA$ for some $g \in \mathrm{GL}(r, \mathbb{Q})$ with integer entries and define T' to be the toric stack $T' = T(I, B)$. Then T is isomorphic to T' as toric varieties.*

Proof. We give an explicit and set theoretic proof. T and T' are defined by

$$T = (\mathbb{C}^n - V(I))/G_A \qquad , \qquad T' = (\mathbb{C}^n - V(I))/G_B \quad ,$$

where $G_A \cong G_B \cong (\mathbb{C}^*)^r$. If we denote $A = (a_{ij})$ and $B = (b_{ij})$, then for $(\lambda_1, \ldots, \lambda_r) \in G_A$ and $(\gamma_1, \ldots, \gamma_r) \in G_B$, the actions are the following:

$$G_A: \qquad (\lambda_1, \ldots, \lambda_r).(x_1, \ldots, x_n) \mapsto (\prod_{i=1}^{r} \lambda_i^{a_{i1}} x_1, \ldots, \prod_{i=1}^{r} \lambda_i^{a_{in}} x_n)$$

$$G_B: \qquad (\gamma_1, \ldots, \gamma_r).(x_1, \ldots, x_n) \mapsto (\prod_{i=1}^{r} \gamma_i^{a_{i1}} x_1, \ldots, \prod_{i=1}^{r} \gamma_i^{a_{in}} x_n)$$

Let (x) and (y) be two vectors in $\mathbb{C}^n$. Let us denote by (x) $\sim_A$ (y) if (x) and (y) are in the same orbit of the action by G_A, and similarly for (x) $\sim_B$ (y). The aim is to show

$$\text{(x)} \sim_A \text{(y)} \quad \text{if and only if} \quad \text{(x)} \sim_B \text{(y)} \quad .$$

If (x) $\sim_B$ (y), then there exists $(\gamma_1, \ldots, \gamma_r) \in (\mathbb{C}^*)^r$ such that

$$(y_1, \ldots, y_n) = (\prod_{i=1}^{r} \gamma_i^{b_{i1}} x_1, \ldots, \prod_{i=1}^{r} \gamma_i^{b_{in}} x_n) \quad .$$

To prove (x) $\sim_A$ (y), we must find $(\lambda_1, \ldots, \lambda_r) \in (\mathbb{C}^*)^r$ such that

$$(y_1, \ldots, y_n) = (\prod_{i=1}^{r} \lambda_i^{a_{i1}} x_1, \ldots, \prod_{i=1}^{r} \lambda_i^{a_{in}} x_n) \quad .$$

This follows from $b_{ij} = \sum_k g_{ik} a_{kj}$, if we put $\lambda_i = \gamma_1^{g_{i1}} \ldots \gamma_r^{g_{ir}}$.
Proof for the only if part is very similar and it is done by replacing g by g^{-1}. □

The result of Lemma 2.3 shows that the expression $T(I, A)$ is not uniquely determined from A, when considered as varieties, and it varies up to the action of a subset of $\mathrm{GL}(r, \mathbb{Q})$. In fact failure of this set to be a subgroup is the problem of well-formedness. In the rest of this section, we complete our task of finding a well formed model for $T(I, A)$. Furthermore, it will be noted that such a model is unique up to $\mathrm{SL}^*(r, \mathbb{Z})$, the group of integer matrices with determinant ± 1.

Definition 2.4. Let $M \in \mathcal{M}_{r \times n}(\mathbb{Z})$ be a rank r matrix ($r < n$). Suppose $m_1, \ldots, m_s$ are all the non-zero $r \times r$ minors of M and let $d_M = \gcd(|m_1|, \ldots, |m_k|)$. The matrix M is called *standard* if $d_M = 1$.

Lemma 2.5. *For any rank r matrix $M \in \mathcal{M}_{r \times n}(\mathbb{Z})$, there exist matrices $g \in \mathrm{GL}(r, \mathbb{Q}) \cap \mathcal{M}_{r \times r}(\mathbb{Z})$ and $N \in \mathcal{M}_{r \times n}(\mathbb{Z})$ such that $M = gN$ and N is a standard matrix of rank r.*

We try to remove every factor of d_M by multiplying M with a matrix whose inverse is in $\mathrm{GL}(r, \mathbb{Q}) \cap \mathcal{M}_{r \times r}(\mathbb{Z})$. Taking the resulting matrix at the end and applying the reverse process completes the proof.

Proof. If $d_M = 1$, then there is nothing to prove. Assume p is a prime factor of d_M and m is the biggest integer for which $p^m \mid d_M$. If p^k (for some positive k) divides every entry of the first row of M then multiply M with an $r \times r$ diagonal matrix $H = (h_{ij})$ with $h_{ii} = 1$ for $i > 1$ and $h_{11} = \frac{1}{p^k}$. It is obvious that $M^{(1)} = HM \in \mathcal{M}_{r \times n}(\mathbb{Z})$ and $d_M = p^k d_{M^{(1)}}$.
If $k = m$ we have managed to remove p^m as it was promised. Now assume $k < m$ and let $M^{(1)} = (a_{ij})$. There is at least one non-zero entry in the first row of $M^{(1)}$ which is not divisible by p. Without loss of generality we can assume this entry is a_{11}. By assumption, there is another non-zero entry in the first row. Assume, without loss of generality, that a_{21} is non-zero and suppose $\gcd(a_{11}, a_{21}) = a$, then there exist integers b and c such that $ba_{11} + ca_{21} = a$. Let H_1 be the following matrix:

$$\begin{pmatrix} b & c & 0 & \cdots & 0 \\ -\frac{a_{21}}{a} & \frac{a_{11}}{a} & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The matrix $M^{(2)} = H^1 M^{(1)}$ has the following shape

$$\begin{pmatrix} a & * & \cdots \\ 0 & * & \cdots \\ * & * & \\ \vdots & & \ddots \end{pmatrix} \quad .$$

Obviously $\det(H^1) = 1$ and a is not divisible by p.

By repeating this process for all entries of the first column, except a_{11}, we can replace them by 0. Now let $M_1^{(2)}$ be the $(r-1)\times(n-1)$ sub-matrix of $M^{(2)}$ obtained by removing the first row and column. Obviously $\det(M^{(2)}) = a.\det(M_1^{(2)})$. This forces p^{m-k} to divide $\det(M_1^{(2)})$.

We can repeat the algorithm above and remove all powers of p from the first row of $M_1^{(2)}$. If there is any factor of p left, we apply the process above to the second column of the new matrix to make its entries equal to zero.

By repeating this algorithm we find a matrix M' for which $d_M = p^m \times d_{M'}$. All these can be done again for a prime factor of $d_{M'}$. After finitely many steps we will have a matrix N with $d_N = 1$. □

Corollary 2.6. *For any $T(I,A)$, there exists a standard matrix B such that $T(I,A) \cong T(I,B)$.*

Proof. This follows from Lemma 2.3 and Lemma 2.5. □

Proposition 2.7. *Let $A \in \mathcal{M}_{r\times n}$ be a matrix of rank r and $T(I,A)$ the corresponding variety as before. The following are equivalent.*

(i) $A\colon \mathbb{Z}^n \to \mathbb{Z}^r$ is surjective.
(ii) $\wedge^r A\colon \wedge^r \mathbb{Z}^n \to \wedge^r \mathbb{Z}^r \cong \mathbb{Z}$ is surjective.
(iii) A is standard.

Definition 2.8. Let A be a standard $r\times n$ matrix with integer entries. Suppose A_k is an $r\times(n-1)$ matrix obtained by removing the k-th column of A. The matrix A is called *well formed* if every A_k $(1 \le k \le n)$ is standard.

Lemma 2.9. *Let $T(I,A)$ be a toric stack defined by an irrelevant ideal I and an $r\times n$ matrix $A = (a_{ij})$. Assume $q \neq 1$ is a positive integer such that $q \mid a_{1j}$ for $j > 1$ but $q \nmid a_{11}$. Define the matrix $B = (b_{ij})$ by $b_{i1} = q.a_{i1}$ and $b_{ij} = a_{ij}$ for $j > 1$. Then $T(I,A) \cong T(I,B)$ as varieties.*

Proof. Pick an ample divisor D such that

$$T = \operatorname{Proj} R_D \quad ,$$

where R is the Cox ring of T generated by $x_1,\dots,x_n$ with degrees correspond to C_i, columns of the matrix A, and D has degree $D = \sum \alpha_i C_i$, where α_i are non-negative integers. Note that we associate D with its degree D and use the same notation for both.

The ring R_{qD} consists of invariant sections of multiples of qD, i.e.

$$R_{qD} = \left(\bigoplus_{j=0}^{\infty} H^0(\mathbb{C}^d, \mathcal{L}^j_{qD})\right)^G \qquad \text{and} \qquad \operatorname{Proj} R_D \cong \operatorname{Proj} R_{qD} \quad .$$

Let $x_1^{a_1}\dots x_n^{a_n}$ be a monomial in R_{qD}. There is a positive integer m such that

$$a_1C_1 + \cdots + a_nC_n = mqD \qquad .$$

In particular, $a_1a_{11} = q\alpha$ for some non-zero integer α, so q divides a_1. Therefore x_1 appears in R_{qD} only to the q^{th} power as x_1^q. Hence

$$R_{qD} \cong \bigoplus_{j=0}^{\infty} H^0(\operatorname{Spec}(\mathbb{C}[x_1^q, x_2, \dots, x_n]), \mathcal{L}^j_D)^G \qquad ,$$

but this is the coordinate ring of $T(I,B)$. □

Definition 2.10. A variety defined by $T(I,A)$ is called *well formed* if its defining matrix A is well formed.

Corollary 2.11. *For any variety $T(I,A)$ there exists a well formed model $T(I,B)$ such that $T(I,A) \cong T(I,B)$.*

Remark 2. $T(I,B)$ is the coarse moduli space for a toric Deligne-Mumford stack $T(I,A)$, where B is the standard matrix for which $T(I,A) \cong T(I,B)$. This has been theoretically studied in [13] and our method treats well-formedness in an explicit way.

Example 2.12. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

This matrix is not standard as $d_A = 2$. The standard (well formed) model can be obtained by multiplying the following 2×2 matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \mathrm{SL}^*(2, \mathbb{Z})$$

and then removing the factor of 2 from the second row, which results in

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The geometry of this example: Consider the weighted projective space $\mathbb{P} = \mathbb{P}(1,1,1,2)$. This is an orbifold with a terminal cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$. Consider eigencoordinates x, y, z, t on $\mathbb{P}$. The projective space $\mathbb{P}$ is covered by 4 open affine patches, three of them are $U_x \cong U_y \cong U_z \cong \mathbb{C}^3$, where U_x, for example, is the Zariski open subset $x \neq 0$, and the fourth patch is $U_t = C^3 /\!\!/ \mathbb{Z}_2$. The action of Z_2 on $\mathbb{C}^3$ is given by $(\bar{x}, \bar{y}, \bar{z}) \mapsto (\epsilon\bar{x}, \epsilon\bar{y}, \epsilon\bar{z})$, for ϵ a second root of unity, and is traditionally denoted by $\frac{1}{2}(1,1,1)$.

Let us explain the toric structure of $\mathbb{P}$. The fan consists of 4 rays:

$$r_1 = (1,0,0), \quad r_2 = (0,1,0)$$
$$r_3 = (0,0,1), \quad r_4 = (-1,-1,-2)$$

and they generate 4 maximal cones:

$$C_1 = \langle r_1, r_2, r_3 \rangle, \ C_2 = \langle r_1, r_3, r_4 \rangle, \ C_3 = \langle r_1, r_2, r_4 \rangle \text{ and } C_4 = \langle r_1, r_2, r_3 \rangle$$

one can associate y, z, t, x with r_1, r_2, r_3 and r_4, in that order, and check the structure of the cones, in particular the fact that C_1, C_2 and C_4 are smooth and C_3 is the singular cone. In order to resolve this singularity it is enough to introduce a new ray $r_5 = (0,0,-1)$ and do the corresponding subdivision. One can compute the Cox ring of this new toric variety (using techniques in [12] and [11]) to see that this variety is the quotient of $\mathbb{C}^5 - Z(I)$ by $(\mathbb{C}^*)^2$, where I, the irrelevant ideal, is $I = (x_1, x_2, x_3) \cap (x_4, x_5)$ when $x_1, \ldots, x_5$ are the eigencoordinates on $\mathbb{C}^5$, and the action is induced by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In other words, this rank 2 toric variety is the weighted blow up of $\mathbb{P}(1,1,1,2)$ at the singular point. For a reader interested in minimal model theory of toric varieties: if one requires to carry out the 2-ray game (as in the Sarkisov programme [7]), starting from this blow up, will see (using techniques in [4]) that the other end results in a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.

3. Blow ups of low rank Cox rings and Sarkisov links

We begin this section by considering a special class of rank 2 toric varieties. The goal is to understand their singularities and constructing tools to explicitly write down the Cox rings of their (toric, weighted) blow ups. In [4] it was shown how the Cox data of a rank 2 Cox ring changes as one runs the type III or IV Sarkisov link. The aim in this section is to understand what happens, in terms of Cox data, as one runs type I or II Sarkisov links.

Definition 3.1. A *weighted bundle over* $\mathbb{P}^n$ is a rank 2 toric variety (or stack) $\mathcal{F} = T(A, I)$ defined by

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m]$.
(ii) The irrelevant ideal of $\mathcal{F}$ is $I = (x_0, \dots, x_n) \cap (y_0, \dots, y_m)$.
(iii) and the $(\mathbb{C}^*)^2$ action on $\mathbb{C}^{n+m+2}$ is given by

$$A = \begin{pmatrix} 1 & \dots & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_m \\ 0 & \dots & 0 & 1 & a_1 & \dots & a_m \end{pmatrix} \quad ,$$

where ω_i are non-negative integers and $\mathbb{P}(1, a_1, \dots, a_m)$ is a weighted projective space.

We denote this quotient by $\mathcal{F}_{\mathbb{P}}(\omega + 0, \dots, \omega_m)$ or sometimes simply by $\mathcal{F}$, when there is no ambiguity.

The following lemma is an easy consequence of our assumptions.

Lemma 3.2. *The weighted bundle $\mathcal{F}$ defined in Definition 3.1 is well formed if and only if the weighted projective space $\mathbb{P}(1, a_1, \dots, a_m)$ is well formed.*

Remark 3. Without loss of generality we assume that any weighted bundle $\mathcal{F}$ that appears in this section is well formed unless otherwise stated.

The following lemma constructs the fan associated to the weighted bundle in Definition 3.1.

Theorem 3.3. *Let $\beta_1, \dots, \beta_m, \alpha_1, \dots, \alpha_n$ be the standard basis of $\mathbb{Z}^{n+m}$. Suppose α_0 and β_0 are the following vectors in $\mathbb{Z}^{n+m}$.*

$$\beta_0 = -\sum_{i=1}^{m} a_i \beta_i \quad , \quad \alpha_0 = -\sum_{j=1}^{n} \alpha_j + \sum_{i=0}^{m} \omega_i \beta_i \quad ,$$

where ω_i are non-negative integers. Let $\sigma_{rs} = \left\langle \beta_0, \dots, \hat{\beta_r}, \dots, \beta_m, \alpha_0, \dots, \hat{\alpha_s}, \dots, \alpha_n \right\rangle$ be the cone in $\mathbb{Z}^{n+m}$ generated by $\beta_0, \dots, \hat{\beta_r}, \dots, \beta_m$ and $\alpha_0, \dots, \hat{\alpha_s}, \dots, \alpha_n$, where α_s and β_r are omitted. If we denote Σ for the fan in $\mathbb{Z}^{n+m}$ generated by maximal cones σ_{rs} for all $0 \le r \le n$ and $0 \le s \le m$, then $\mathcal{F} \cong T(\Sigma)$.

Proof. We compute the GIT construction of this fan following the recipe of Cox given in [11] §10. By assumption, rays $\alpha_0, \dots, \alpha_n, \beta_0, \dots, \beta_m$ in $N = \mathbb{Z}^{m+n}$ form $\Delta(1)$, the set of 1-dimensional cones in Σ. Let us associate the variables $x_0, \dots, x_n, y_0, \dots, y_m$ to these rays. For a given maximal cone σ, define x^σ to be the product of all variables not coming from edges of σ. But maximal cones in Σ are exactly σ_{rs}, which immediately implies $x^{\sigma_{rs}} = x_s y_r$. The irrelevant ideal is given by

$$I = (x^\sigma \mid \sigma \in \Sigma \text{ is a maximal cone}) = (x_s y_r \mid 0 \le s \le n \text{ and } 0 \le r \le r) \quad .$$

It is clear that the primary decomposition of this ideal is $I = (x_0, \dots, x_n) \cap (y_0, \dots, y_m)$.

In order to describe the GIT construction of $T(\Sigma)$ we must find the group G such that

$$T(\Sigma \cong (\mathrm{Spec}[x_0, \dots, x_n, y_0, \dots, y_m] - V(I))/G \quad .$$

The group $G \subset (\mathbb{C}^*)^{m+n+2}$ is defined by

$$G = \{(\mu_0, \dots, \mu_n, \lambda_0, \dots, \lambda_m) \in (\mathbb{C}^*)^{m+n+2} \mid \prod_{i=0}^{n} \mu_i^{\langle e_k, \alpha_i \rangle} . \prod_{j=0}^{m} \lambda_j^{\langle e_k, \beta_j \rangle} = 1, \text{ for all } k\} \quad ,$$

where $e_1, \dots, e_{m+n}$ form the standard basis of $\mathbb{Z}^{m+n}$. But the standard basis of $\mathbb{Z}^{m+n}$, by assumption, is $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$.

Computing this set implies that $(\mu_0, \dots, \mu_n, \lambda_0, \dots, \lambda_m) \in G$ if and only if

$$\mu_i . \mu_0^{\langle \alpha_0, \alpha_i \rangle} . \lambda_0^{\langle \beta_0, \alpha_i \rangle} = 1 \quad \text{and} \quad \lambda_j . \mu_0^{\langle \alpha_0, \beta_j \rangle} . \lambda_0^{\langle \beta_0, \beta_j \rangle} = 1 \quad .$$

In other words, λ_0 and μ_0 determine all other λ_j and μ_i. Therefore the group G is isomorphic to $(\mathbb{C}^*)^2$ and the action on coordinate variables is defined by

$$((\mu, \lambda).x_0) = \mu x_0 \qquad ((\mu, \lambda).x_i) = \mu^{-\langle \alpha_0, \alpha_i \rangle} \lambda^{-\langle \beta_0, \alpha_i \rangle} x_i \quad ,$$

$$((\mu, \lambda).y_0) = \lambda y_0 \qquad ((\mu, \lambda).y_j) = \mu^{-\langle \alpha_0, \beta_j \rangle} \lambda^{-\langle \beta_0, \beta_j \rangle} y_j \quad .$$

In other words, $(\mathbb{C}^*)^2$ acts on $\mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m]$ by the matrix

$$A = \begin{pmatrix} 1 & \dots & 1 & 0 & \omega_0 a_1 - \omega_1 & \dots & \omega_0 a_m - \omega_m \\ 0 & \dots & 0 & 1 & a_1 & \dots & a_m \end{pmatrix} \quad .$$

We have shown so far that $T(\Sigma) \cong T(A, I)$. Multiplying A on the left by the matrix

$$\begin{pmatrix} 1 & -\omega_0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}^*(2, \mathbb{Z}) \quad ,$$

together with Lemma 2.3 proves that $\mathcal{F} \cong T(\Sigma)$. □

Remark 4. In [20] Chapter 2, Reid gives a detailed analysis of rational scrolls, which, in our setting, are the weighted bundles over $\mathbb{P}^1$, with weights 1 only. In fact these are the smooth weighted bundles.

Proposition 3.4. *A well formed weighted bundle $\mathcal{F}$, defined in Definition 3.1, is covered by $(n+1)(m+1)$ patches, each of them isomorphic to a quotient of $\mathbb{C}^{n+m}$ by a cyclic group $\mathbb{Z}_r$, for some positive integer r.*

Proof. We construct the patches $\mathcal{U}_{ij}$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. Note that in the toric level, $\mathcal{U}_{ij} = (x_i y_j \neq 0)$ corresponds to the maximal cone σ_{ij} as in Proposition 3.3.

$$\mathcal{U}_{ij} = \operatorname{Spec} \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m, x_i^{-1}, y_j^{-1}]^{\mathbb{C}^* \times \mathbb{C}^*}$$

Computing the invariants gives

$$\mathcal{U}_{ij} = \operatorname{Spec} \mathbb{C}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_0^{a_j}}{y_j} . x_i^{\omega_0 a_j - \omega_j}, \dots] \quad .$$

Again powers of x_i appear to make each term invariant under the action of the first coordinate of $(\mathbb{C}^*)^2$ and each y_k comes with a power that is the first number which is 0 modulo a_j. In other words, these invariants are exactly the same as those of $\frac{1}{a_j}(0, \dots, 0, 1, a_1, \dots, a_n)$. □

3.1. **Blow ups of weighted projective space.** Example 2.12 already explained the blow up of the weighted projective space $\mathbb{P}(1,1,2)$ at its singular point. In general the rank two toric variety T (or stack, if not already well-formed) defined by the homogeneous coordinate ring $C[y,x_0,\dots,x_n]$ and the irrelevant ideal $I=(y,x_0,\dots,x_k)\cap(x_{k+1},\dots,x_n)$ and the weight system (indicating the action of $(\mathbb{C}^*)^2$)

$$\begin{pmatrix} \alpha & 0 & \dots & 0 & -b_{k+1} & \dots & -b_n \\ 0 & a_0 & \dots & a_k & a_{k+1} & \dots & a_n \end{pmatrix} \qquad ,$$

for $1\le k\le n-2$ is the (weighted) blow up of the centre $X:(x_{k+1}=\dots=x_n=0)\cong\mathbb{P}(a_0,\dots,a_k)\subset\mathbb{P}(a_0,\dots,a_n)$. Details of this constructions are left to the reader to check. A more complicated situation is explained in the next part and the idea and techniques of the proofs there can be applied to this case.

The birational geometry of this space follows the variation of GIT as explained in [4]. As an illustration, the following example shows how this can be used to study birational geometry of hypersurfaces of weighted projective spaces. Consider the Fano 3-fold X defined by an equation of degree 24 in $\mathbb{P}=\mathbb{P}(1,1,6,8,9)$, in other words, in a suitable coordinate system, X is the vanishing of $f=x_5^2x_3+x_4^3+x_3^4+\dots+x_1^{24}$ in $\mathbb{P}$. Suppose $p_5=(0:0:0:0:1)$. It is easy to check that the germ near $p_5\in\mathbb{P}$ is of type $1/9(1,1,6,8)$ and the germ near $p_5\in X$ is a terminal quotient singularity of type $1/9(1,1,8)$. Consider T the blow up of this point, with respect to above description, given by a rank two variety with weight system

$$\begin{pmatrix} 3 & 0 & -2 & -6 & -1 & -1 \\ 0 & 9 & 8 & 6 & 1 & 1 \end{pmatrix}$$

with the coordinate system $\mathbb{C}[u,x_5,x_4,x_3,x_2,x_1]$ and the irrelevant ideal $I=(u,x_5)\cap(x_4,x_3,x_2,x_1)$. In other words the blow up is given by $(x_5,x_4u^{\frac{2}{3}},x_3u^2,x_2u^{\frac{1}{3}},x_1u^{\frac{1}{3}})$, in coordinates. The restriction of this toric construction to X indicates the blow up $\hat{X}\to X$ that corresponds to the $1/3(1,1,2)$ blow up of $p_5\in X$. One can check that T is not well-formed and its well-formed model has the weight system

$$\begin{pmatrix} 1 & 3 & 2 & 0 & 0 & 0 \\ 0 & 9 & 8 & 6 & 1 & 1 \end{pmatrix}$$

with respect to which, the equation of $\hat{X}$ has bi-degree $(6,24)$, with equation $x_3x_5^2+x_4^3+x_3^4u^6+\dots+u^6x_1^3$. Following the 2-ray game of the ambient space (using techniques of [4]), it is easy to verify that $\hat{X}$ forms a fibration over $\mathbb{P}(1,1,6)$ with elliptic fibres $E_6\subset\mathbb{P}(1,2,3)$. In [6], Cheltsov and Park study elliptic, and K3, fibrations of Fano 3-folds by considering a projection and looking at the local equation of the fibres. As shown in the example above their results can be recovered using our methods by means of global calculations.

3.2. **Blow ups of rank 2 toric varieties.** Now we construct Cox rings of rank 3, obtained by blowing up some centres in a rank 2 toric variety. Then we try to understand the nature of the maps from these varieties to the rank 2 ones. We do this on weighted bundles over $\mathbb{P}^1$, i.e. when the coordinate ring is $\mathbb{C}[x_0,x_1,y_0,\dots,y_n]$ with irrelevant ideal $I=(x_0,x_1)\cap(y_0,\dots,y_n)$ and weight system

$$\begin{pmatrix} 1 & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_m \\ 0 & 0 & 1 & a_1 & \dots & a_m \end{pmatrix} \qquad ,$$

for positive integers $a_1,\dots,a_n$ and non-negative integers $\omega_0,\dots,\omega_n$.

It was shown in Proposition 3.4 that each germ $p_{rs}\in\mathcal{U}_{rs}$ defined by $x_i=y_j=0$, for all $i\neq r$ and $j\neq s$, has a cyclic quotient singularity of type $\frac{1}{a_j}(0,1,\dots,a_m)$. Of course this singularity is not isolated. However, instead of blowing up the orbifold locus, we blow up a closed point. The reason for doing this blow up is that we often want to consider the blow up of some subvarieties of $\mathcal{F}$ only at this particular point, see next section for an illustration. We do this by considering the blow up of the ambient space at this point and restrict our attention to the subvariety under this blow up.

Fix $k \in \{0, \dots, m\}$ and let T be a rank 3 toric variety defined by

(i) $\mathrm{Cox}(T) = \mathbb{C}[X_0, X_1, Y_0, \dots, Y_m, \xi]$,

(ii) the irrelevant ideal

$$J = (X_0, X_1) \cap (Y_0, \dots, Y_m) \cap (\xi, X_1) \cap (\xi, Y_k) \cap (X_0, Y_0, \dots, \hat{Y_k}, \dots, Y_m) \quad \text{and}$$

(iii) the action of $(\mathbb{C}^*)^3$ given by the matrix

$$\begin{pmatrix} 1 & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_{k-i} & -\omega_k & -\omega_{k+1} & \dots & -\omega_m & 0 \\ 0 & 0 & 1 & a_1 & \dots & a_{k-1} & a_k & a_{k+1} & \dots & a_m & 0 \\ b_k & 0 & b_0 & b_1 & \dots & b_{k-1} & 0 & b_{k+1} & \dots & b_m & -a_k \end{pmatrix} \quad ,$$

where $b_0, \dots, b_m$ are strictly positive integers such that

$$b_i \equiv a_i \mod a_k \text{ for } i \neq k \qquad \text{and} \qquad b_k = ra_k \text{ for some positive integer } r \quad .$$

Proposition 3.5. *The rank 3 toric variety T constructed above is the blow up of the weighted bundle $\mathcal{F}$ over $\mathbb{P}^1$ in Definition 3.1 at the point* $(0:1;0:\cdots:0:1)$.

Proof. By Proposition 3.3, the fan associated to $\mathcal{F}$ consists of 1-dimensional cones β_0, β_1 and $\alpha_0 \dots, \alpha_m$ in $N = \mathbb{Z}^{m+1}$ with $2m+2$ maximal cones

$$\sigma_{0i} = \langle \beta_1, \alpha_0 \dots, \hat{\alpha_i}, \dots, \alpha_m \rangle \text{ and } \sigma_{1j} = \langle \beta_0, \alpha_0 \dots, \hat{\alpha_j}, \dots, \alpha_m \rangle \text{ for } 0 \leq i, j \leq m \quad .$$

The last row of the defining matrix of T is clearly adding a new ray in the cone σ_{0k}. The fact that the generator of this ray is an integral vector in N is guaranteed by the conditions imposed on b_i. This implies that T is the blow up of F at a point if it has the correct irrelevant ideal. We complete the proof by showing the irrelevant ideal of the Cox ring of this toric blow up is precisely the ideal of T. This is done by taking the subdivision of σ_{0k} and computing the irrelevant ideal of the new fan using the method of [11], as in the proof of Proposition 3.3. The fan of this blow up of Σ consists of rays $\beta_0, \beta_1, \alpha_0, \dots, \alpha_m, \gamma$ and maximal cones

$$\sigma'_{ki} = \langle \beta_0, \alpha_0 \dots, \hat{\alpha_i}, \dots, \hat{\alpha_k}, \dots, \alpha_m, \gamma \rangle \text{ for } i \neq k \text{ and } \sigma'_k = \langle \alpha_0 \dots, \hat{\alpha_k}, \dots, \alpha_m, \gamma \rangle$$

coming from the subdivision of σ_{0k} together with the remaining cones σ_{ij}. If we associate the new variable ξ to the ray γ and X_0, X_1 to β_0, β_1 and Y_i to α_i, then the irrelevant ideal of this toric variety is the ideal generated by

$$X_1 \cdot Y_i \cdot Y_k \text{ for all } i \neq k, \quad X_0 \cdot X_1 \cdot Y_k, \quad X_1 \cdot Y_i \cdot \xi \text{ for all } i \neq k, \quad X_0 \cdot Y_i \cdot \xi \text{ for all } i \quad .$$

The primary decomposition of this ideal is the irrelevant ideal of T. □

4. Proof of the main results

Our initial model X is defined as a complete intersection of two hypersurfaces in $\mathcal{F}$, where $\mathcal{F}$ is a $\mathbb{P}^4$-bundle over $\mathbb{P}^1$. If we denote by $y_0, y_1, x_0, \cdots, x_4$, the global coordinates on $\mathcal{F}$, then the Cox ring of $\mathcal{F}$ is $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[y_0, y_1, x_0, \cdots, x_4]$ with weights

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -3 & -3 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad .$$

and irrelevant ideal $I = (y_0, y_1) \cap (x_0, \cdots, x_4)$.

4.1. **Cox ring and description of the initial model.** Define Q_1 to be the vanishing of the polynomial f of bi-degree $(-3, 2)$, and similarly $Q_2 = (g = 0)$, where $\deg(g) = (-2, 2)$. Assume furthermore that f has no monomial term x_0x_4 and similarly g has no $y_0x_0x_3$ or $y_1x_0x_3$, and otherwise f and g are general.

Lemma 4.1. *$X = Q_1 \cap Q_2 \subset \mathcal{F}$ is smooth.*

Proof. A simple calculation on the Jacobian matrix of X shows that $\mathrm{Sing}(X) \subset \mathbb{P}^1_0$, where $\mathbb{P}^1_0 = (x_1 = x_2 = x_3 = x_4 = 0)$. Having monomials of type $x_0 \times l$, where l is a linear term, in the equation of X imply smoothness along this line. $\square$

Proposition 4.2. $\mathrm{Cox}(X) = \dfrac{\mathrm{Cox}(\mathcal{F})}{(f, g)}$.

Proof. It is easy to check, using methods in [1] Section 4.2 that $\mathrm{Pic}(X) \cong \mathrm{Pic}(\mathcal{F})$. Factoriality of $\mathcal{F}$ and Lemma 4.1 imply that $X \subset \mathcal{F}$ is a neat embedding, see [16] Definition 2.5. The result follows from [16] Corollary 2.7. $\square$

4.2. **Different Mori structures.** Now we show how the other Mori fibre space models, birational to X, are obtained.

By rules of Sarkisov programme, see [8] §2.2, there are two possibilities for the start of the programme: having a Mori fibre space $X \to S$, either run one step of the MMP on S, obtain $S \dashrightarrow T$, and consider $\mathrm{Pic}(X/T)$ or do a blow up on X, obtain $Z \to X$, and consider $\mathrm{Pic}(Z/S)$. In both cases the relative Picard group has rank 2. One generator of this group corresponds to $Z \to X$ or $S \dashrightarrow T$ and the other generator indicates the beginning of the so-called 2-ray game, and in correct setting the Sarkisov link. See [8] for details. Running the first type is most commonly used for varieties constructed in similar ways to our objects, see for example [1,3]. The general setting for this kind of links is described in [4]. Following these techniques one can see that X is birational to Y, a del Pezzo fibration of degree 2 over $\mathbb{P}^1$. The link between X and Y consists of an anti-flip, of local type $(1, 1, -1, -3)$ followed by an Atiyah flop.

Proposition 4.3. *X is birational to a del Pezzo fibration of degree 2 over $\mathbb{P}^1$.*

The Cox ring of Y is $\mathrm{Cox}(Y) = \mathbb{C}[u, v, x, y, z, t, s]/(f, g)$, with weights

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 \end{pmatrix},$$

and irrelevant ideal $I_Y = (u, v) \cap (x, y, z, t, s)$.

Note that, for simplicity, we have renamed the variables, i.e., the variables u, x, y, z, t, s are exactly $x_4, x_3, x_2, x_1, x_0, y_0, y_1$, in that order. The weight matrix is that of $\mathcal{F}$ in opposite order, in rays and columns, multiplied by a matrix A, where

$$\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix} \in \mathrm{SL}^*(2, \mathbb{Z}).$$

The equations of f and g must be easily read after the substitution, in particular, $f = 0.vz + uz + x^2t + \cdots$ and $g = 0.vzt + 0.vzs + y^2 + xz + \cdots$, with bi-degrees $(-1, 3)$ and $(-2, 4)$, with respect to the new weights. By Proposition 3.4, $\mathcal{F}_1$, the ambient toric variety in which Y is embedded, has two lines of singularities of types $\mathbb{A}^1 \times 1/2(1, 1, 1, 1)$ and $\mathbb{A}^1 \times 1/3(1, 1, 1, 2)$. It is easy to check that Y does not meat the first line and it intersects the second line at the point $p_{vz} = (u = x = y = t = s = 0)$. In particular, p_{vz} is a terminal singularity of type $1/3(1,1,2)$, as one can eliminate the variables u and x in an analytical neighbourhood of this point using f and g.

We want to show that a blow up of the point p_{vz} is the start of a Sarkisov link on Y. By a result of Kawamata [18] this blow up is a unique weighted blow up of type $(1,1,2)$ with discrepancy $\frac{1}{3}$. In other words, if we denote the blow up of Y by $\mathcal{Y}$, and the exceptional divisor of the blow up by E, then

$$K_{\mathcal{Y}} = K_Y + \frac{1}{3}E$$

Define a rank 3 toric Cox ring by $\mathcal{R} = \mathbb{C}[u,v,x,y,z,t,s,w]$, the irrelevant ideal $\mathcal{I} = (u,v) \cap (x,y,z,t,s) \cap (u,x,y,t,s) \cap (w,v) \cap (w,z)$ and weights given by the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 3 & 0 & a & 2 & 0 & 1 & 1 & -3 \end{pmatrix}.$$

By Proposition 3.5, $\mathcal{T} = T(\mathcal{I}, \mathcal{A})$ is a blow up of $\mathcal{F}_1$, for a positive integer $a = 3k+1$. The aim is to show that for a particular k, $\mathcal{Y}$ is neatly embedded in $\mathcal{T}$.

Note that the blow up map is given by

$$\varphi((u,v,x,y,z,t,s,w)) \mapsto (uw, v, w^{\frac{a}{3}}x, w^{\frac{2}{3}}y, z, w^{\frac{1}{3}}t, w^{\frac{1}{3}}s)$$

It is easy to check that $\mathcal{A}$ is not well formed. In order to use the adjunction formula for subvarieties of $\mathcal{T}$ we need to consider the well formed model. By techniques of Section 2 one can verify that the well formed model is

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 1 & 0 & k & 0 & -1 & 0 & 0 & -1 \end{pmatrix}.$$

Lemma 4.4. *For $a = 4$, $\mathcal{Y} \longrightarrow Y$ is the Kawamata blow up of the point $p_{vz} \in Y$, where $\mathcal{Y} \subset \mathcal{T}$ is the birational transform of Y by φ.*

Proof. Using adjunction formula $-K_Y \sim \mathcal{O}_Y((0,1))$. If we denote by $\tilde{f}$ and $\tilde{g}$ the defining equations of $\mathcal{Y} \in \mathcal{T}$, then $\deg \tilde{f} = (-1,3,0)$ and $\deg \tilde{g} = (-2,4,0)$, hence, using adjunction formula again, we have $-K_{\mathcal{Y}} \sim \mathcal{O}_{\mathcal{Y}}((0,1,k-1))$. In other words, $K_{\mathcal{Y}} \sim -D_x - E$, where $D_x = (x = 0)$ is a principal divisor on $\mathcal{Y}$ and $E = (w = 0)$ is the exceptional divisor. On the other hand, $\varphi^*(K_Y) \sim -D_x - \frac{a}{3}E$. Using Kawamata's condition that the discrepancy is equal to $\frac{1}{3}$, we conclude that $a = 4$. $\square$

Lemma 4.5. *Y is square birational to a degree 2 del Pezzo model neatly embedded in a toric variety as a hypersurface.*

Proof. Using toric MMP we can see that the 2-ray game on $\mathcal{T}$, associate to the rank 2 relative Picard group of $\mathcal{T}/\mathbb{P}^1_{u:v}$, consists of a flop to $\mathcal{T}'$ followed by a divisorial contraction to a rank 2 toric variety $\mathcal{F}_2$. The flop is the contraction of the fibring lines in a $\mathbb{P}^1$-bundle of $\mathbb{P}(1,1,2)$ on one side and extracting another $\mathbb{P}^1$-bundle on the other side of the flop. In terms of Cox rings, this is the replacement of $\mathcal{I}$ by another ideal $\mathcal{I}' = (u,v) \cap (w,v) \cap (u,x) \cap (w,y,z,t,s) \cap (x,y,z,t,s)$. If we rewrite $\mathcal{A}$, after a $\mathrm{SL}^*(3,\mathbb{Z})$ modification, as

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 & -4 & -1 & -1 & -1 \end{pmatrix}.$$

By Proposition 3.5, $\mathcal{T}' \longrightarrow \mathcal{F}_2$ is a divisorial contraction with exceptional divisor $E' = (u = 0)$ to a point, where $\mathrm{Cox}(\mathcal{F}') = \mathbb{C}[w, v, z, y, x, t, s]$, with irrelevant ideal $I_2 = (w, v) \cap (z, y, x, t, s)$ and the weights

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 3 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

We can write down the equation of this contraction by

$$\psi((u, v, x, y, z, t, s, w)) \mapsto (uw, v, u^4 z, u^2 y, x, ut, us),$$

in particular, the image of the contraction is the point $p_{vx} \in \mathcal{F}_2$.

Restricting all these to $\mathcal{Y}$, we see that the flop (of the ambient space) corresponds to a flop of 6 lines, associate to the intersection of a quartic (coming from the birational transform of g) and a cubic (coming from f) in $\mathbb{P}(1, 1, 2)$, that is 6 points by Lemma 9.5 in [17]. The exceptional divisor is a cubic in $\mathbb{P}(1, 1, 2)$. The birational transforms of f and g in $\mathcal{F}_2$ are, respectively $\hat{f} = z + wxy + vyt + \cdots$ and $\hat{g} = y^2 + xz + \cdots$. In particular, we can eliminate z globally, and consider Z, the birational transform of Y, embedded as a hypersurface in a rank 2 toric variety with Cox ring equal to that of $\mathcal{F}_2$, with z eliminated. Z is the vanishing of the polynomial $y^2 = vyxt + w^2 t^4 + \cdots$. It is easy to check that Z is a fibration of degree 2 del Pezzo surfaces over $\mathbb{P}^1_{v:w}$, and is square birational to $Y/\mathbb{P}^1$. ☐

Lemma 4.6. *Z is birational to a conic bundle.*

Proof. This appears as Family 6 in Theorem 3.3 of [1]. ☐

The following diagram shows the geometry of X and its birational models.

$$\begin{tikzcd}
 & & & \mathcal{Y} \arrow[dl, "\text{blow up}"'] \arrow[r, dashed, "6\times(1,1,-1,-1)"] & \mathcal{Y}' \arrow[dr, "\text{blow up}"] & & \\
X \arrow[r, dashed, "(1,1,-1,-3)"] \arrow[d, "dP_4"'] & X_1 \arrow[r, dashed, "(1,1,-1,-1)"] & Y \arrow[d, "dP_2"] & & & Z \arrow[r, "\cong"] \arrow[d, "dP_2"'] & Z \arrow[d, "\text{conic bundle}"] \\
\mathbb{P}^1 & & \mathbb{P}^1 & & & \mathbb{P}^1 & \mathbb{P}^2
\end{tikzcd}$$

In order to complete the proof of Theorem 1.3 we need to show that X is not rational. We use the result of Alexeev [2] to obtain this.

Theorem 4.7 ([2], Theorem 2)**.** *If the Euler characteristic of a (standard) del Pezzo fibration of degree 4 over $\mathbb{P}^1$ does not belong to $\{0, -4, -8\}$, then it is not rational.*

Lemma 4.8. *X is not rational.*

Proof. By the same calculation as [14] Example 3.2.11 we can compute $\chi(X) = -28$, the Euler characteristic of X. The result follows from Theorem 4.7. ☐

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