

Linear orbits of plane curves

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OVERVIEW

$GL(n+1, \mathbb{C}) = \{\text{invertible } (n+1) \times (n+1) \text{ } \mathbb{C}\text{-matrices}\}$
acts on $V = \mathbb{C}^{n+1}$

$PGL(n+1, \mathbb{C})$

$= \{\text{invertible } (n+1) \times (n+1) \text{ matrices up to scalar}\}$

acts on $\mathbb{P}V$.

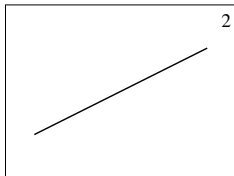
Make it act on $\mathbb{P} \text{Sym}^d V^\vee = \text{space of degree-}d \text{ hypersurfaces.}$

Definition

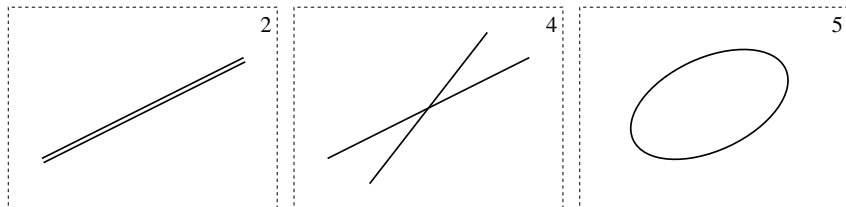
The **linear orbit** of X is its orbit under $PGL(n+1)$.

Examples, $n = 2$:

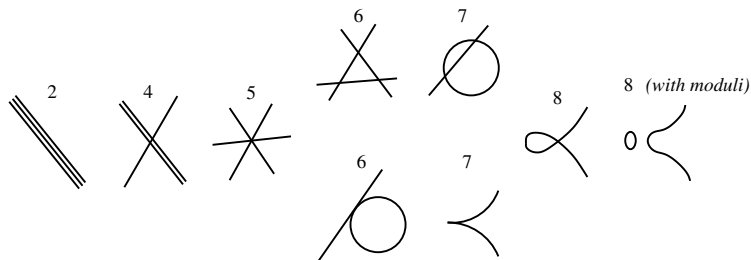
$d = 1$:



$d = 2:$



$d = 3:$



$d \geq 4$: ??

Probably description of set of orbits known for $d = 4$;
and probably out of reach for $d \geq 5$.

Natural question:

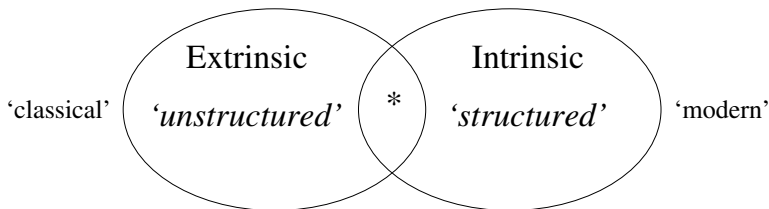
Given hypersurface $X \subseteq \mathbb{P}^n$, 'describe' its $\mathrm{PGL}(n+1)$ -orbit.

'describe':

- dimension
- degree of closure
- singularities of closure
- behavior in families
- ...

We will focus on the degree of the orbit closure.

Motivation: Enumerative geometry



'Intrinsic': *Gromov-Witten invariants*

Example: *How many plane curves of degree d and geometric genus g pass through suitably many ($= 3d + g - 1$) general points?*

Severi degrees

(Schubert. . . Ran. . . Caporaso-Harris. . . Kontsevich. . .)

Structure: *Quantum cohomology*; also *Fock space approach*,
Cooper-Pandharipande '12

Main character: moduli space.

'Extrinsic' example: *How many smooth plane curves of degree d are tangent to suitably many general lines?*

Characteristic numbers

(Schubert. . . Fulton-MacPherson. . . Kleiman-Speiser . . . Vakil. . .)

For $d = 1, 2, 3, 4, \dots$: 1, 1, 33616, 23011191144 (Vakil, '98),
wide open for $r \geq 5$!

Are these numbers 'structured' by something like quantum cohomology? (I don't know.)

Main character: humble projective space.

Question: What's both extrinsic *and* intrinsic?

One possible answer

(*) *how many smooth plane curves with given degree d and moduli class contain suitably many general points?*

I.e.: Given an abstract curve C , in how many ways can it be realized as a plane curve of given degree so as to contain N general points? (N = dimension of parameter space.)

(Structure?? 'Isotrivial GW invariants'?)

Example

\mathbb{P}^{14} = space of plane quartics (genus= 3). Have rational map

$$\mathbb{P}^{14} \supseteq \text{general } \mathbb{P}^6 \dashrightarrow \mathcal{M}_3$$

What is the degree of this map? This is just *one* question (*).

Fair question: Can't quantum cohomology methods do this?

Answer: Yes, for small genus. **Zinger**, early 2000's. ($g \leq 3$?)

Arbitrary genus/degree?

Fact:

For $d \geq 4$, $C \subseteq \mathbb{P}^2$ smooth curve, the answer to (*) equals the degree of the $\mathrm{PGL}(3)$ -orbit closure of C .

So (*) is a special case of the general invariant theory question stated at the beginning.

Theorem (—, C. Faber)

Explicit formula for the degree of the $\mathrm{PGL}(3)$ -orbit closure of a smooth plane curve of degree d :

$$\frac{(\text{polynomial in } d) - (\text{contribution from special flexes})}{\# \text{ of automorphisms}}$$

Example

of realizations of given C genus g , deg. d smooth $\ni N$ points:

d	g	N	#
1	0	2	1
2	0	5	1
3	1	8	12 ($j \neq 0, 1728$) 4 ($j = 0$) 6 ($j = 1728$)

Example

d	g	N	#
4	3	8	$\frac{14280 - \text{contribution from hyperflexes}}{ \text{Aut} }$

So there are 14280 ways to realize a general genus-3 curve as a plane quartic containing 8 given general points.

Example

For a general sextic curve C (genus = 10), the numerator is $1119960 = 2^3 \cdot 3^3 \cdot 5 \cdot 17 \cdot 61$. Consequence: $|\text{Aut}(C)|$ cannot be a multiple of 7 or 11.

But note that there exist genus 10 curves with 14, 22, 33 automorphisms.

Main point here:

Question (*) was formulated for smooth plane curves.
However, its invariant-theoretic formulation makes sense for
arbitrary plane curves, and in fact arbitrary projective hypersurfaces.

C : plane curve, equation $F(x_0 : x_1 : x_2) = 0$, $\deg F = d$.

$\varphi \in \mathrm{PGL}(3) \rightsquigarrow C \circ \varphi$, equation $F(\varphi(x_0 : x_1 : x_2)) = 0$,
 translate of C .

$O_C := \mathrm{PGL}(3)\text{-orbit} = \{\text{translates of } C\} \subseteq \mathbb{P}^{\frac{d(d+3)}{2}}$

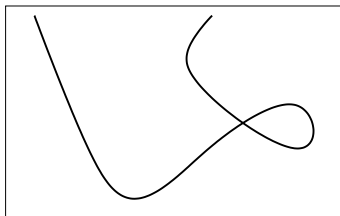
Main question

$$\deg \overline{O_C} = ?$$

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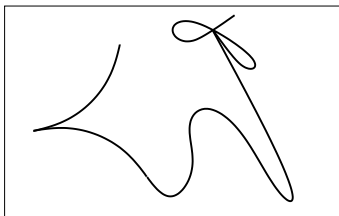
$$\deg \overline{O_C} = ?$$

This question makes sense irrespective of singularities of C .
Notice the difference w.r.t. *moduli of map* viewpoint:



- $\dim\{\text{nodal quartics}\} = 13$; (QC, genus 2)
- $\dim\{\text{realizations of fixed genus-2 } C \text{ as nodal quartic}\} = 10$;
- $\dim O_C = \dim \text{PGL}(3) = 8$ for a quartic C .

But also:



From a moduli-of-map perspective, need to record the special points, *plus* local data specifying the singularities.

I don't know how to do this.

(It can be done for special singularities, e.g., cusps.

Spectacular enumerative results by **Nguyen**.)

No such complication from the projective geometry point of view.

Theorem (—, C. Faber)

A procedure computing $\deg \overline{O_C}$ for arbitrary C .

The answer depends on

- *degree and multiplicities of components of C ;*
- *degree of closure of stabilizer of C (as a subvariety of \mathbb{P}^8);*
- *information about ‘special points’ of C :*
 - *inflection points of support;*
 - *tangent cone at singularities;*
 - *Puiseux pairs of branches at singular points.*

Remark

Popov has posed the problem of computing the degree of orbit closures of arbitrary actions of linear groups.

The theorem solves this problem for $\mathrm{PGL}(3)$ acting on $\mathbb{P}(\mathrm{Sym}^d \mathbb{C}^3)$.

Example

Enumerative applications to **characteristic numbers**:

Degree for C = general sextic: 1119960.

Contribution of a cusp: 23544.

Hence $\deg \overline{O_C} = (1119960 - 9 \cdot 23544) / (\# \text{Stab}(C))$
for a sextic with 9 cusps.

Fact: $\# \text{Stab}(C) = 18$; so $\deg \overline{O_C} = \frac{908064}{18} = 50448$.

Dually: *The number of smooth cubics with fixed j -invariant ($\neq 0, 1728$) tangent to 8 lines in general position is 50448.*

Similar:

$\#$ nodal cubics tangent to 8 lines in gen. pos. = $\deg \overline{O_C}$ for dual C ,
i.e., quartic with three cusps: $\frac{14280 - 3 \cdot 3960}{6} = 400$ (Schubert).

The approach

Fix C , degree d ; $N = \frac{d(d+3)}{2}$.

Compactify $\mathrm{PGL}(3)$ to \mathbb{P}^8 , and resolve the indeterminacies of the extended action map: π proper birational,

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}
 \quad \varphi \in \mathbb{P}^8 \mapsto \alpha(\varphi) := C \circ \varphi$$

Note: α is not defined at $\varphi \in \mathbb{P}^8$ such that $\mathrm{im} \varphi \subseteq C$.

For example, if C has no linear components, then the base locus of α is $\mathbb{P}^2 \times C$ (=rk-1 matrices whose image is a point of C).

$\overline{O_C} = \overline{\mathrm{im} \alpha}$. Want: class of $\overline{O_C}$ in \mathbb{P}^N , i.e., $c_1(\mathcal{O}(1))^{\dim \overline{O_C}} \cap \overline{\mathrm{im} \alpha}$.

Right question:

$$\pi_*(ch(\tilde{\alpha}^* \mathcal{O}(1)) \cap [V]) = 1 + a_1 H + \cdots + a_8 \frac{H^8}{8!} = ?$$

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

Definition (Adjusted predegree polynomial ('a.p.p.')

$$\pi_*(ch(\tilde{\alpha}^* \mathcal{O}(1)) \cap [V]) = 1 + a_1 H + \cdots + a_8 \frac{H^8}{8!}$$

Lemma

- $\dim O_C (= 8 - \dim \mathrm{Stab}(C)) = \text{degree } r \text{ of the a.p.p.};$
- $\deg \overline{O_C} = \frac{a_r}{\deg \overline{\mathrm{Stab} C}}.$

Proof: Just chase definitions.

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

How to construct V , compute a.p.p.?

Balancing act:

- ① $V =$ closure of graph of α fits the diagram; but very singular, hard to control intersection theory.
- ② If $\pi =$ sequence of blow-ups at smooth centers, then intersection theory is doable; but how to find centers?

Strategy:

- Perform (2) when possible, in ‘enough’ cases;
- Then study (1) by degeneration arguments. This means...

Consider

$$\begin{array}{ccc}
 & V = \bar{\Gamma} & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

where $\Gamma \subseteq \mathbb{P}^8 \times \mathbb{P}^N$, graph of α .

I.e., $\bar{\Gamma}$ = blow-up of \mathbb{P}^8 along the base **scheme** S of α .

Definition (PNC)

The **projective normal cone** for C is the inverse image of S in $\bar{\Gamma}$.

$(\varphi, X) \in \text{PNC} \iff$ there exists a germ of a curve $\varphi(t) \subset \mathbb{P}^8$ such that $\varphi(0) = \varphi$, and $X = \lim_{t \rightarrow 0} C \circ \varphi(t)$.

So studying the PNC amounts to understanding all ‘limits of translates’ of the given curve.

Harris-Morrison, *Moduli of curves*, p.138:

Flat completion problem: describe all curves in \mathbb{P}^n that can arise as flat limits of families of curves over the punctured disc together with a line bundle giving embeddings of the curves in the family as curves in \mathbb{P}^n .

Understanding the PNC amounts to solving the ‘isotrivial’ version of this problem, for $n = 2$: all curves in the family are just translations of a fixed curve.

Theorem (—, C. Faber)

Procedure to describe the PNC for a given arbitrary curve C : irreducible decomposition, class and multiplicity of the components.

This is what is needed for the second part of the ‘strategy’ sketched earlier.

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

First part of the strategy: construct $\pi : V \rightarrow \mathbb{P}^8$ explicitly as a sequence of blow-ups at smooth centers.

This can be carried out for special curves: C smooth;

C s.t. $\dim O_C < 8$; some other cases.

- **C smooth curve:**

- Base locus of α : $\mathbb{P}^2 \times C \subseteq \mathbb{P}^8$; blow-up along $\mathbb{P}^2 \times C$;
- blow-up along a \mathbb{P}^1 -subbundle of the exceptional divisor;
- blow-up along \mathbb{P}^2 's over flexes, a number of times depending on the order of the flex.

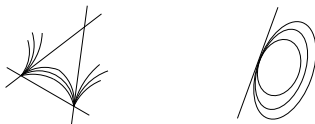
This constructs V explicitly. Intersection theory \rightsquigarrow a.p.p.

- C has 'small orbit': $\dim O_C < 8$, i.e., $\dim \text{Stab } C > 1$.

Theorem (—, C. Faber)

A classification of these curves.

Representative pictures of the most interesting cases:



corresponding to

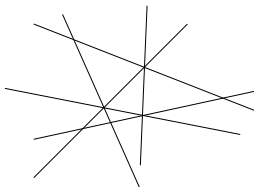
$$x^a y^b z^c \prod_i (z^n + \lambda_i x^m y^{n-m}) \quad (n \geq 3) \quad , \quad x^a \prod_i (z^2 + xy + \mu_i x^2) \quad .$$

Theorem (—, C. Faber)

Explicit construction of $\pi : V \rightarrow \mathbb{P}^8$ for all such curves.

Typically, the blow-ups mirror embedded resolution of the curve.

- C = a line arrangement (with multiplicities)



- $\varphi \in \mathbb{P}^8$ s.t. $\text{im } \varphi$ = an intersection point: a union of \mathbb{P}^2 's;
- $\varphi \in \mathbb{P}^8$ s.t. $\text{im } \varphi \subseteq$ a line in the arrangement: a union of \mathbb{P}^5 's;
- $\pi : V \rightarrow \mathbb{P}^8$ may be obtained by blowing up the \mathbb{P}^2 's and then the proper transforms of the \mathbb{P}^5 .

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

General case:

No good blow-up sequence is known in general.

$\Gamma :=$ graph of α ;

$\bar{\Gamma} \cong$ blow-up of \mathbb{P}^8 along base scheme S of α ;

$E :=$ PNC = exceptional divisor in this blow-up.

Intersection theory:

The a.p.p. is determined by the class $[E]$ in $\mathbb{P}^8 \times \mathbb{P}^N$. We need:

- ① description of the components of E ;
- ② multiplicity of each component;
- ③ class of each component.

(3) is the 'easy' part:

components are 7-dimensional, and union of $\mathrm{PGL}(3)$ -orbits;
so these components can be classified *a priori*, and classes can be related to enumerative geometry of curves with small orbits.

(1)+(2): subjectively *hard*.

For a given C , need to determine all possible limits of translates, and associated multiplicities.

Example

$$C : x^3z^4 - 2x^2y^3z^2 + xy^6 - 4xy^5z - y^7 = 0$$

irreducible, singular at $(1 : 0 : 0)$.

$$\varphi(t) := \begin{pmatrix} 1 & 0 & 0 \\ t^8 & t^9 & 0 \\ t^{12} & \frac{3}{2}t^{13} & t^{14} \end{pmatrix} \quad (\text{Note that } \mathrm{im} \varphi(0) = (1 : 0 : 0).)$$

$\lim_{t \rightarrow 0} C \circ \varphi(t): x^3(8x^2 + 3y^2 - 8xz)(8x^2 - 3y^2 + 8xz) = 0$
reducible, union of two quadritangent conics and triple line.

Summary of description of PNC:

Five different origins of components of PNC of a given C :

- ① linear components of C ;
- ② nonlinear components of C ;
- ③ points at which the tangent cone has ≥ 3 components;
- ④ points with special features of the Newton polygon;
- ⑤ points with special features of Puiseux pairs.

Limits due to (4): unions of cuspidal curves;

limits due to (5): unions of quadritangent conics (cf. example).

'Structure' ??

No organizational principle such as QC. (As far as I know.)

Hints here and there of interesting structure. For example:

Proposition

C : reduced curve, $\deg C > 1$; L : line transversal to C at non-flex points; then $\text{a.p.p.}(C \cup L)$ equals the truncation to H^8 of

$$\text{a.p.p.}(C) \cdot \left(1 + H + \frac{H^2}{2}\right) \cdot \left(1 - \frac{H^6}{24} + \frac{7H^7}{60} - \frac{13H^8}{80}\right)^{\#(C \cap L)}$$

The a.p.p. of a line is $1 + H + \frac{H^2}{2}$, so this result expresses a weak multiplicativity property of a.p.p.'s

If C is a configuration of lines, then the extra 'correction' term is not there(!), so the a.p.p. is multiplicative on the nose in this case. Consequence: the a.p.p. for d general lines is the truncation to H^8 of $\left(1 + H + \frac{H^2}{2}\right)^d$.

Example

The a.p.p. for a triangle C is

$$\left(1 + H + \frac{H^2}{2}\right)^3 = 1 + 3H + \frac{9}{2}H^2 + 4H^3 + \frac{9}{4}H^4 + \frac{3}{4}H^5 + \frac{1}{8}H^6.$$

Therefore $\dim O_C = 6$ (clear), and $\deg \overline{O_C} = (6!/8)/\deg \text{Stab}(C)$.
 $\deg \text{Stab}(C) = 6$ (clear), so $\deg \overline{O_C} = 15$.

Also clear combinatorially: count # of triangles through 6 points.

$$4 \text{ lines: } \left(1 + H + \frac{H^2}{2}\right)^4 = 1 + 4H + \dots + \frac{1}{16}H^8$$

$$\rightsquigarrow \deg \overline{O_C} = \frac{8!}{16 \cdot 4!} = 105.$$

Exercise: do this combinatorially.

But combinatorics does not give $\deg \overline{O_C}$ for ≥ 5 lines:

$$\left(1 + H + \frac{H^2}{2}\right)^5 = 1 + 5H + \dots + \frac{25}{16}H^8 \rightsquigarrow \deg \overline{O_C} = 63000$$

This has actually been worked out in all dimensions.

(**Tzigantchev**: Explicit formulas for plane arrangements in space.)

Explicit formulas are messy in general.

One case when they are reasonably neat: contribution of unbranched singularities.

For example, consider a curve C of degree d , ordinary flexes, and singularities of type (t^m, t^n) , with no further Puiseux pairs. Also assume $\text{Stab } C$ is trivial.

Theorem (—, C. Faber)

The degree of the orbit closure of C is

$$d^8 - \left\{ (1 + dk)^8 \left[\frac{4d^2}{(1 + k)^3(1 + 2k)^3} + \sum_{p \in C \text{ of type } (t^m, t^n)} mn \left(\frac{m^2 n^2}{(1 + mk)^3(1 + nk)^3} - \frac{4}{(1 + k)^3(1 + 2k)^3} \right) \right] \right\}_2$$

(Here $\{\cdot\}_2$ extracts the coefficient of k^2 in the given expression.)

Example

If C is smooth, with ordinary flexes, then it has $3d(d-2)$ points 'of type (t, t^3) '.

According to the theorem, if $\text{Stab } C$ is trivial, then $\deg \overline{O_C}$ equals

$$\begin{aligned}
 & d^8 - \left\{ (1 + dk)^8 \left[\frac{4d^2}{(1+k)^3(1+2k)^3} \right. \right. \\
 & \quad \left. \left. + 3d(d-2) \cdot 3 \left(\frac{9}{(1+3k)^3(1+nk)^3} - \frac{4}{(1+k)^3(1+2k)^3} \right) \right] \right\}_2 \\
 & \quad = d^8 - d(1372d^3 - 7992d^2 + 15879d - 10638) \\
 & \quad = d(d-2)(d^6 + 2d^5 + 4d^4 + 8d^3 - 1356d^2 + 5280d - 5319)
 \end{aligned}$$

C : general plane curve of degree d

d	$\deg \overline{O_C}$
4	14280
5	188340
6	1119960
7	4508280
8	14318256
9	38680740
10	92790480
11	203104440
12	413183160
13	791558196
14	1442049000
15	2516992920
16	4233892320
...	...

DOUBLES

Warm-up: d -tuples in \mathbb{P}^1

Recall: We are interested in the orbits of the action of $\mathrm{PGL}(n+1)$ on the space of hypersurfaces in \mathbb{P}^n of a fixed degree d .

Main interest: $n = 2$. What about $n = 1$?

A d -tuple of points C in \mathbb{P}^1 is the zero-set of a degree d homogeneous polynomial $F(x_0, x_1) \in \mathbb{C}[x_0, x_1]$.

So C consists of points p_1, \dots, p_r with multiplicities m_1, \dots, m_r such that $\sum m_i = d$.

$\{d\text{-tuples}\} = \mathbb{P}^d$: 'coordinates' of a d -tuple are the $(d+1)$ coefficients of F .

$\mathrm{PGL}(2)$ acts on C : $\varphi = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \mapsto$ the d -tuple defined by

$$F(a_{00}x_0 + a_{01}x_1, a_{10}x_0 + a_{11}x_1) \quad .$$

This amounts to translating the points p_i , preserving multiplicities.
 $O_C \subseteq \mathbb{P}^d$: $\mathrm{PGL}(2)$ -orbit, i.e., set of translates. Clearly $\dim O_C \leq 3$.

Question

$$\deg \overline{O_C} = ?$$

First approach: Combinatorics.

Remark: The condition of 'containing a point' is linear in the coefficients of F : each $p \in \mathbb{P}^1$ determines a hyperplane H_p in \mathbb{P}^d .

Thus, if $\dim O_C = 3$, then $\deg \overline{O_C} = \#(H_{p_1} \cap H_{p_2} \cap H_{p_3} \cap O_C)$ for general points p_1, p_2, p_3 and (possibly) counting intersection multiplicities.

Now $C' \in (H_{p_1} \cap H_{p_2} \cap H_{p_3} \cap O_C) \iff C'$ is a $\mathrm{PGL}(2)$ -translate of C , and $p_1, p_2, p_3 \in C'$.

Therefore:

Evident Lemma

If $\dim O_C = 3$:

$$\begin{aligned}\deg \overline{O_C} &= \text{number of } \textit{translates} \text{ of } C \text{ containing } p_1, p_2, p_3 \\ \deg \overline{O_C} &= \frac{\text{number of } \textit{translations} \text{ of } C \text{ containing } p_1, p_2, p_3}{\text{Stab}(C)} \\ &= \frac{\# \text{ of translations of } 0, 1, \infty \text{ to points of } C}{\text{Stab}(C)}\end{aligned}$$

Understood: *counting multiplicities*. Local computation:

- multiplicities = 1 if C consists of d distinct points.
- in general, multiplicity for a translation = product of the multiplicities of images of p_1, p_2, p_3 .

Now:

a $\mathrm{PGL}(3)$ -translation is determined by the images of $0, 1, \infty$.

So the lemma reduces the computation of $\deg \overline{O_C}$ to elementary combinatorics.

Corollary

$C = d$ -tuple of distinct points, trivial stabilizer. Then

$$\deg \overline{O_C} = d(d-1)(d-2)$$

(This argument may already be found in Enriques-Fano.)

Example

If C is a general d -tuple:

- The orbit is 'small' ($\dim < 3$) iff $d = 1, 2$. ($\overline{O_C} = \mathbb{P}^1, \mathbb{P}^2$.)
- If $d = 3$, $\deg \overline{O_C} = \frac{3 \cdot 2 \cdot 1}{3!} = 1$. (And indeed $\overline{O_C} = \mathbb{P}^3$.)
- A general 4-tuple has stabilizer $C_2 \times C_2$: $\deg \overline{O_C} = \frac{4 \cdot 3 \cdot 2}{4} = 6$.
- For $d \geq 5$, a general d -tuple has trivial stabilizer:
 $\deg \overline{O_C} = d(d-1)(d-2)$.

Combinatorics & multiplicity considerations give $\overline{O_C}$ for arbitrary d -tuples.

Example

Double point, $d - 2$ distinct points (otherwise general):



Then $H_0 \cap H_1 \cap H_\infty \cap \overline{O_C}$ consists of

- $(d - 2)(d - 3)(d - 4)$ points, mult. 1; and
- $3(d - 2)(d - 3)$ points, mult. 2.

If $\text{Stab} = \text{trivial}$, $\deg \overline{O_C} = (d + 2)(d - 2)(d - 3)$.

Legitimate question:

Since combinatorics works so well, why try anything else?

Simplest answer: combinatorics doesn't scale to higher dimension.

So let's try to do combinatorics without combinatorics.
Basic diagram:

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(2) \hookrightarrow \mathbb{P}^3 & \xrightarrow{\alpha} & \mathbb{P}^d
 \end{array}
 \quad \varphi \in \mathbb{P}^3 \mapsto \alpha(\varphi) := C \circ \varphi$$

$$\mathbb{P}^3 = \left\{ \varphi = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \right\}, \quad C : F(x_0, x_1) \in \mathbb{P}^d:$$

$$\alpha(\varphi) = C \circ \varphi : F(a_{00}x_0 + a_{01}x_1, a_{10}x_0 + a_{11}x_1).$$

Base scheme?

Lemma

The base locus of α consists of a disjoint union of linearly embedded \mathbb{P}^1 's.

Lemma

The base locus of α consists of a disjoint union of linearly embedded \mathbb{P}^1 's.

In fact, the base scheme of α is isomorphic to $\mathbb{P}^1 \times C$.

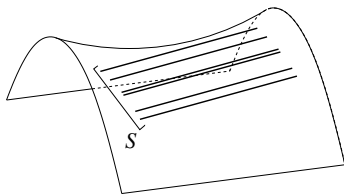
Proof: The base locus of α must be contained in the complement of $\mathrm{PGL}(2)$, i.e., it must consist of $\mathrm{rk} = 1$ matrices:

$$\alpha := \begin{pmatrix} k_0 i_0 & k_1 i_0 \\ k_0 i_1 & k_1 i_1 \end{pmatrix} \in \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

$$\begin{aligned} F(a_{00}x_0 + a_{01}x_1, a_{10}x_0 + a_{11}x_1) &\equiv 0 \iff \\ F(k_0 i_0 x_0 + k_1 i_0 x_1, k_0 i_1 x_0 + k_1 i_1 x_1) &\equiv 0 \iff \\ (k_0 x_0 + k_1 x_1)^d F(i_0, i_1) &\equiv 0 \iff F(i_0, i_1) = 0 \end{aligned}$$



Base scheme: supported on $S = \mathbb{P}^1 \times C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$:



$\text{rk} = 1$ matrices with free kernel, image in C .

Theorem

α is resolved by blowing up \mathbb{P}^3 along the support of $\mathbb{P}^1 \times C$.

Proof: Focus on one component of $\mathbb{P}^1 \times C$, say $\mathbb{P}^1 \times \{p\}$,
 p of multiplicity r in C .

WLOG, $F(x_0, x_1) = x_1^r G(x_0, x_1)$ with $x_1 \nmid G$.

$V := \text{blow-up of } \mathbb{P}^3 \text{ along the support of } S$.

$F(x_0, x_1) = x_1^r G(x_0, x_1)$. Work near $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$:

$\left\{ \varphi = \begin{pmatrix} 1 & a \\ b & c \end{pmatrix} \right\}$, blow-up locus with local equations $b = c = 0$.

Chart in blow-up V : $\left\{ \begin{pmatrix} 1 & a \\ \tilde{b}e & e \end{pmatrix} \right\}$, $e = 0$ exceptional divisor.

Lift of $\alpha : \mathbb{P}^3 \dashrightarrow \mathbb{P}^d$ to blow-up:

$$\begin{aligned} \tilde{\alpha} \left(\begin{pmatrix} 1 & a \\ \tilde{b}e & e \end{pmatrix} \right) &= (\tilde{b}ex_0 + ex_1)^r G(x_0 + ax_1, \tilde{b}ex_0 + ex_1) \\ &= e^r (\tilde{b}x_0 + x_1)^r G(x_0 + ax_1, \tilde{b}ex_0 + ex_1) \\ &\equiv (\tilde{b}x_0 + x_1)^r G(x_0 + ax_1, \tilde{b}ex_0 + ex_1) \\ &\neq 0 \end{aligned}$$

So $\tilde{\alpha} : V \rightarrow \mathbb{P}^2$ is defined everywhere. □

$$\begin{array}{ccccc}
 & E & \hookrightarrow & V & \\
 & \swarrow & & \searrow & \\
 & & \pi & & \\
 S & \hookrightarrow & \mathbb{P}^3 & \xrightarrow{\alpha} & \mathbb{P}^d
 \end{array}$$

E = exceptional divisor = $E_1 \cup \dots \cup E_m$, one connected component for each point of C .

h = hyperplane class in \mathbb{P}^d .

Evident Lemma

If $\dim O_C = 3$, then $\deg \overline{O_C} = \frac{\int \tilde{\alpha}^*(h)^3}{\deg \alpha}$.

Definition

The **predegree** of C is the intersection number $\int \tilde{\alpha}^*(h)^3$.

Definition

The **predegree** of C is the intersection number $\int \tilde{\alpha}^*(h)^3$.

Why stop there? Consider all powers $\tilde{\alpha}^*(h)^i$.

With $H :=$ hyperplane class in \mathbb{P}^3 , $\pi_*(\tilde{\alpha}^*(h)^i) = a_i H^i$ for some $a_i \in \mathbb{Z}$.

Definition

The **predegree polynomial** of C is $\sum_i \pi_*(\tilde{\alpha}^*(h)^i) = \sum_i a_i H^i$.

Evident Lemma

- The degree of the predegree polynomial equals $\dim O_C$;
- For all C , $\deg \overline{O_C} = \frac{a_{\dim O_C}}{\deg \overline{\text{Stab}(C)}}$.

Another point of view:

$$\alpha : \mathbb{P}^3 \dashrightarrow \mathbb{P}^d;$$

$$\Gamma \subseteq \mathbb{P}^3 \times \mathbb{P}^d: \text{ graph.}$$

$\bar{\Gamma} \cong$ the blow-up of \mathbb{P}^3 along the base scheme S of α .

Note that $A_*(\mathbb{P}^3 \times \mathbb{P}^d)$ is generated by $H := \pi^*(H)$, $h := \tilde{\alpha}^*(h)$.

Lemma

$$[\bar{\Gamma}] = \sum_i a_i H^i h^{d-i}.$$

Proof:

If V is *any* variety resolving α , then have proper bir. $V \rightarrow \bar{\Gamma}$.

Projection formula:

$$\sum_i \pi_*(\tilde{\alpha}^*(h)^i) = \sum_i a_i H^i \iff a_i = H^{3-i} h^i \cdot [\bar{\Gamma}].$$

□

So predegree polynomial \rightsquigarrow 'set h to 1 in $[\bar{\Gamma}]$ '.

Plan: **to extract information from $\bar{\Gamma} = \text{Bl}_S \mathbb{P}^3$.**

Definition

Let $p = (p_0 : p_1) \in \mathbb{P}^1$. The **point condition** determined by p is the hypersurface X_p of \mathbb{P}^3 defined by $F(a_{00}p_0 + a_{01}p_1, a_{10}p_0 + a_{11}p_1)$.

Remarks:

- The base scheme of α is the (scheme-theoretic) intersection of all point-conditions: $S = \bigcap_{p \in \mathbb{P}^1} X_p$.
- What earlier proof showed is that for all p

$$\pi^*(X_p) = r_1[E_1] + \cdots + r_m[E_m] + [\tilde{X}_p] \quad ,$$

where r_i is the multiplicity of the i -th point of C , and

- \tilde{X}_p = the proper transform of X_p , $[\tilde{X}_p] = \tilde{\alpha}^*(h)$.
- ... and further $\bigcap_p \tilde{X}_p = \emptyset$, so that $\tilde{\alpha}$ is defined everywhere,
- $\pi^{-1}(S) = \bigcap_{p \in \mathbb{P}^1} \pi^{-1}(X_p) = r_1 E_1 + \cdots + r_m E_m$.

(Remember this!)

Interlude: Segre classes

$S \subseteq T$ schemes $\rightsquigarrow s(S, T) \in A_* S$, the *Segre class* of S in T .

Characterized by:

- If S is regularly embedded in T , $s(S, T) = c(N_S T)^{-1} \cap [S]$.
- If $f : T' \rightarrow T$ is proper birational: $s(S, T) = f_* s(f^{-1}(S), T')$:
Birational invariance of Segre classes.

Example

- $s(\mathbb{P}^1, \mathbb{P}^3) = c(N_{\mathbb{P}^1 \mathbb{P}^3})^{-1} \cap [\mathbb{P}^1] = [\mathbb{P}^1] - 2[\mathbb{P}^0]$.
- D Cartier divisor $\rightsquigarrow s(D, V) = D - D^2 + D^3 - \dots$.
- $T' = \text{blow-up of } T \text{ along } S$, $E = \text{exceptional divisor}$,
$$s(S, T) = f_* s(E, T') = f_*(E - E^2 + E^3 - \dots)$$

Fact:

If S is the base scheme of a rational map $\mathbb{P}^r \dashrightarrow \mathbb{P}^d$, then $s(S, \mathbb{P}^r)$ carries essentially the same information as the class of $\bar{\Gamma}$ in $\mathbb{P}^r \times \mathbb{P}^d$.

Segre classes \leftrightarrow predegree polynomials

Notation:

\mathcal{L} : line bundle on a variety; $\gamma = \sum_j \gamma^{(j)}$ a Chow class, indexed by codimension.

Definition

$$\gamma \otimes \mathcal{L} := \sum_j c(\mathcal{L})^{-j} \cap \gamma^{(j)} .$$

This is an action of Pic! $\gamma \otimes (\mathcal{L} \otimes \mathcal{M}) = (\gamma \otimes \mathcal{L}) \otimes \mathcal{M}$.

Theorem

Let $\iota : S \hookrightarrow \mathbb{P}^3$ be the base scheme of the rational map α associated with a d -tuple C . Then the predegree polynomial of C equals

$$\frac{([\mathbb{P}^3] - \iota_* s(S, \mathbb{P}^3)) \otimes \mathcal{O}(-dH)}{1 - dH}$$

Proof: Not now.

Now? You were told to remember

- $\pi^{-1}(S) = \cap_{p \in \mathbb{P}^1} \pi^{-1}(X_p) = r_1 E_1 + \cdots + r_m E_m.$

The E_i are disjoint, so

$$\begin{aligned} s(r_1 E_1 + \cdots + r_m E_m, V) &= \sum_i s(r_i E_i, V) \\ &= \sum_i (r_i E_i - r_i^2 E_i^2 + r_i^3 E_i^3 - \cdots) \quad . \end{aligned}$$

Birational invariance:

$$s(S, \mathbb{P}^3) = \sum_i \pi_* (r_i E_i - r_i^2 E_i^2 + r_i^3 E_i^3) \quad .$$

How do we evaluate this push-forward?

$$E_i = \pi^{-1}(\mathbb{P}^1), \text{ so } \pi_* s(E_i, V) = s(\mathbb{P}^1, \mathbb{P}^3) = [\mathbb{P}^1] - 2[\mathbb{P}^0].$$

$$\pi_*(E_i - E_1^2 + E_i^3) = [\mathbb{P}^1] - 2[\mathbb{P}^0].$$

Therefore:

$$\pi_*(r_i E_i - r_i^2 E_i^2 + r_i^3 E_1^3) = r_i^2 [\mathbb{P}^1] - 2r_i^3 [\mathbb{P}^0].$$

We have proven:

Lemma

If C is a d -tuple of points with multiplicities r_1, \dots, r_m , then

$$\iota_* s(S, \mathbb{P}^3) = \left(\sum_i r_i^2 \right) [\mathbb{P}^1] - 2 \left(\sum_i r_i^3 \right) [\mathbb{P}^0] \quad .$$

Corollary

Let C be a d -tuple of points with multiplicities r_1, \dots, r_m . Let $r^{(j)} = \sum_i r_i^j$ (so $r^{(1)} = d$). Then the predegree polynomial of C is

$$1 + dH + (d^2 - r^{(2)})H^2 + (d^3 - 3dr^{(2)} + 2r^{(3)})H^3$$

Proof:

Relation between Segre classes and predegree polynomials: the predegree equals

$$\frac{([\mathbb{P}^3] - \iota_* s(S, \mathbb{P}^3)) \otimes \mathcal{O}(-dH)}{1 - dH} = \frac{([\mathbb{P}^3] - r^{(2)}[\mathbb{P}^1] + 2r^{(3)}[\mathbb{P}^0]) \otimes \mathcal{O}(-dH)}{1 - dH}$$

The effect of $\otimes \mathcal{O}(-dH)$ is to divide terms of codimension j by $(1 - dH)^j$, therefore this equals

$$\left(\frac{1}{1 - dH} - \frac{r^{(2)}H^2}{(1 - dH)^3} + \frac{2r^{(3)}H^3}{(1 - dH)^4} \right) \cap [\mathbb{P}^3]$$

Taylor \rightsquigarrow given expression.



Predegree polynomial

$$C = r_1 p_1 + r_2 p_2 + \cdots + r_m p_m$$

$$r^{(j)} := \sum_i r_i^j$$

Predegree polynomial

$$1 + dH + (d^2 - r^{(2)})H^2 + (d^3 - 3dr^{(2)} + 2r^{(3)})H^3.$$

Example

$$r_1 = 2, r_2 = \cdots = r_{d-2} = 1$$



$$r^{(2)} = 4 + (d-2) = d+2; \quad r^{(3)} = 8 + (d-2) = d+6;$$
$$\text{predegree} = d^3 - 3d(d+2) + 2(d+6) = (d-2)(d-3)(d+2).$$

Adjusted predegree polynomial (a.p.p.)

Alternative formulation:

$e_i := i$ -th elementary function on the multiplicities r_i (so $e_1 = d$).

Predegree polynomial

$$1 + e_1 H + 2 e_2 H^2 + 6 e_3 H^3.$$

Proof: Nothing to prove! □

With this in mind, the next definition is unavoidable. Recall that the predegree polynomial really is

$$\sum_i \pi_* (\tilde{\alpha}^*(c_1(\mathcal{O}(1)))^i) \cap [V] \quad .$$

Definition

The **adjusted predegree polynomial** (a.p.p.) of C is

$$\sum_i \frac{\pi_*(\tilde{\alpha}^*(c_1(\mathcal{O}(1)))^i) \cap [V]}{i!} = \pi_*(ch(\tilde{\alpha}^* \mathcal{O}(1)) \cap [V]) \quad .$$

Summarizing:

Theorem

Let C be a tuple of points p_i , with multiplicities r_i . Then

$$a.p.p.(C) = \prod_i (1 + r_i H)$$

(truncated to H^3).

Proof:

Let e_i be the elementary symmetric functions in the multiplicities.

Then predegree polynomial $= 1 + e_1 H + 2 e_2 H^2 + 6 e_3 H^3$,

therefore $a.p.p.(C) = 1 + e_1 H + e_2 H^2 + e_3 H^3$

$=$ truncation of $\sum e_j H^j = \prod_i (1 + r_i H)$.



Example

C : d -tuple of distinct points:

$$\text{a.p.p.}(C) = (1 + H)^d = 1 + dH + \binom{d}{2}H^2 + \binom{d}{3}H^3$$

so predegree $= 6! \binom{d}{3} = d(d-1)(d-2)$.

'Combinatorics explained'

Remarkable fact:

This result generalizes to arbitrary dimensions:

Theorem

Let X be a simple normal crossing divisor consisting of d hyperplanes in \mathbb{P}^n . Then

$$\text{a.p.p.}(X) = \left(1 + H + \frac{H^2}{2} + \cdots + \frac{H^n}{n!} \right)^d$$

truncated to $H^{(n+1)^2-1}$.

One can in fact even allow multiplicities: if the i -th hyperplane appears with multiplicity r_i , then

$$\text{a.p.p.}(X) = \prod_i \left(1 + r_i H + \frac{r_i^2 H^2}{2} + \cdots + \frac{r_i^n H^n}{n!} \right)$$

Proof: Blow-up sequence \rightsquigarrow computation of Segre class of base locus. □

One more reformulation:

Theorem

Let C be a tuple of points p_i , with multiplicities r_i , $\sum r_i = d$. Then

$$\text{a.p.p.}(C) = e^{dH} \prod_i \left(1 - \frac{r_i^2}{2} H^2 + \frac{r_i^3}{3} H^3 \right)$$

(truncated to H^3).

Proof:

$$\begin{aligned} e^{-dH} \text{a.p.p.}(C) &= e^{-dH} \left(1 + dH + \frac{(d^2 - r^{(2)})}{2} H^2 + \frac{(d^3 - 3dr^{(2)} + 2r^{(3)})}{3!} H^3 \right) \\ &= 1 - \frac{r^{(2)}}{2} H^2 + \frac{r^{(3)}}{3} H^3 + \dots \end{aligned}$$

splits as a product mod H^4 .



Interpretation:

e^{dH} : 'global' contribution.

$1 - \frac{r^2 H^2}{2} + \frac{r^3 H^3}{3}$: 'local' *multiplicative* contr. of a mult. r point.

This also has a counterpart in higher dimension.

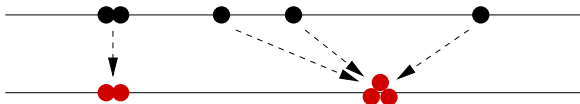
$$\begin{array}{ccc} & V & \\ \pi \swarrow & & \searrow \tilde{\alpha} \\ \mathbb{P}^3 & \xrightarrow{\alpha} & \mathbb{P}^d \end{array}$$

Now that we have constructed V , what else can we do with it?

- Study the **boundary** $\partial O_C := \overline{O_C} \setminus O_C$ of an orbit:
If $d \geq 3$, $\partial O_C = \tilde{\alpha}(E)$, image of exceptional divisor.

Boundary of orbit closure

If C is a d -tuple, ∂O_C consists of the orbit closures of 2-uples $x^r y^{d-r}$, where r is the multiplicity of a point on C .



The way to think about this:

- Consider a 'germ' ('arc'?) $\tilde{\gamma}(t)$ of a smooth curve centered at a point of the exceptional divisor in V ;
- Map down to \mathbb{P}^3 : $\gamma(t) := \pi(\tilde{\gamma}(t)) \in \text{PGL}(2)$ for $t \neq 0$;
- Determine $\lim_{t \rightarrow 0} C \circ \gamma(t)$.

Example

$$C: y^2(x^3 - y^3) = 0; \gamma(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

$$C \circ \gamma(t): t^2 y^2 (x^3 - t^3 y^3)$$

$$\lim_{t \rightarrow 0} C \circ \gamma(t): y^2 x^3$$

Studying the exceptional divisor 'is the same as' classifying flat **limits** of families of projectively equivalent d -tuples.

For d -tuples, one can access the exceptional divisor directly. To scale to higher dimension, one will have to deal directly with limits.

Boundary and multiplicities

$$\begin{array}{ccc} & V & \\ \pi \swarrow & & \searrow \tilde{\alpha} \\ \mathbb{P}^3 & \xrightarrow{\alpha} & \mathbb{P}^d \end{array}$$

What else?

How singular are these orbit closures?

Fact: $S \subseteq T$ subvariety; then multiplicity of T along S
= coefficient of S in Segre class $s(S, T)$. Therefore:

Lemma

$(d \geq 3)$ If $C' \in \overline{O_C}$, then

$$\text{mult}_{C'} \overline{O_C} = \frac{\int s(\tilde{\alpha}^{-1}(C'), V)}{\# \text{Stab}(C)} .$$

Numerator: **premultiplicity**.

Only interesting for $C' \in \partial O_C$; hence again involving exceptional divisor.

In general, $\overline{O_C}$ is singular along all components of ∂O_C .

Example

C : general d -tuple, $d \geq 5$. (So Stab is trivial.)

One boundary component, $x^2 y^{d-2}$.

Then $\text{mult}_{\partial O_C} \overline{O_C} = 2d$. ('Each point contributes 2.')

Even for these boundary components, the premultiplicity is subtle: the contribution of $p \in C$ depends on the **Hessian** of the residual tuple C_p .

'Most singular' points of $\overline{O_C}$: d -fold points, i.e. O_{x^d} .

Fact: O_{x^d} is cut out scheme-theoretically by vanishing of Hessians. So may evaluate $\text{mult}_{x^d} \overline{O_C}$ by pulling back Hessians to V .

Example

C : general d -tuple, $d \geq 5$. (So Stab is trivial.)

Then $\text{mult}_{x^d} \overline{O_C} = 6(d - 2)$.

Can an orbit closure be **smooth**? Yes!

Need $\text{mult}_{x^d} \overline{O_C} = 1$, i.e., premultiplicity = order of stabilizer.

The end-result is very pretty. . .

'Visualize' a d -tuple:

$\mathbb{P}^1\mathbb{C} =$ Riemann sphere;

d -tuple \leftrightarrow vertices on the sphere \leftrightarrow polygons/polyhedra.

A d -tuple is *simple* if it consists of d distinct points.

Smoothness I

The smooth orbit closures of simple d -tuples, $d \geq 3$, correspond to the *regular triangulations* of the sphere:

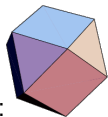
- the equilateral triangle
- the tetrahedron
- the octahedron
- the icosahedron

One can also characterize smoothness *in codimension 1*, i.e., along ∂O_C .

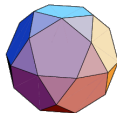
Smoothness II

The orbit closure of a simple d -tuple is smooth in codimension 1 if and only if it corresponds to a *quasi-regular* polyhedron (in the sense of Coxeter), i.e., one of

- the regular polygons
- the cube
- the dodecahedron
- the cuboctahedron
- the icosidodecahedron



Cuboctahedron:



Icosidodecahedron:

Predegree of $\overline{O_C}$, last word

Complicated expression for the predegree of $\overline{O_C}$:

$$d^3 - \left\{ (1+dk)^3 \left[\frac{d}{(1+k)^2} + \sum_{p \in C \text{ of type } (t^m)} m \left(\frac{m}{(1+mk)^2} - \frac{1}{(1+k)^2} \right) \right] \right\}_1$$

C : d -tuple, $\dim O_C = 3$. $\{\cdot\}_1$: coefficient of k^1 .

Compare with the predegree of $\overline{O_C}$ for a degree d plane curve C with ordinary flexes and singularities of type (t^m, t^n) , no further Puiseux pairs:

$$d^8 - \left\{ (1+dk)^8 \left[\frac{4d^2}{(1+k)^3(1+2k)^3} + \sum_{p \in C \text{ of type } (t^m, t^n)} mn \left(\frac{m^2 n^2}{(1+mk)^3(1+nk)^3} - \frac{4}{(1+k)^3(1+2k)^3} \right) \right] \right\}_2$$

A pattern extending to arbitrary dimension?

SMOOTH CURVES

The case of curves

$C \subseteq \mathbb{P}^2$: curve, degree d :

$C = V(F)$, $F(x_0, x_1, x_2)$ homogeneous, degree d .

$\mathrm{PGL}(3)$ acts on C :

$\varphi \in \mathrm{PGL}(3) \rightsquigarrow C \circ \varphi := F(\varphi(x_0, x_1, x_2)) \in \mathbb{P}^{d(d+3)/2}$.

$O_C := \text{orbit}$, $\overline{O_C} := \text{orbit closure}$. Main focus: $\deg \overline{O_C}$.

For d -tuple of points, had combinatorial option.

Combinatorics won't help here.

The other approach:

$$\begin{array}{ccc} & V & \\ \pi \swarrow & & \searrow \tilde{\alpha} \\ \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^{d(d+3)/2} \end{array} \quad \varphi \in \mathbb{P}^8 \mapsto \alpha(\varphi) := C \circ \varphi$$

$$\overline{O_C} = \overline{\mathrm{im} \alpha}.$$

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^{d(d+3)/2}
 \end{array}$$

Recall—More natural object of study:

$$\pi_*(ch(\tilde{\alpha}^* \mathcal{O}(1)) \cap [V]) \in A_* \mathbb{P}^8 = \mathbb{Z}[H]/(H^9)$$

(adjusted) predegree polynomial of C .

- Same information as class of $\bar{\Gamma} \subseteq \mathbb{P}^8 \times \mathbb{P}^{d(d+3)/2}$, closure of the graph of α .
- Same information as the **Segre class** in \mathbb{P}^8 of the base scheme S of α .
- Therefore, more easily accessible if π is a sequence of blow-ups at smooth centers.

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^{d(d+3)/2}
 \end{array}$$

Remark: V resolves the indeterminacies of α

$\iff V$ dominates the closure $\bar{\Gamma}$ of the graph of α .

Keep in mind: $\bar{\Gamma} = \mathrm{Bl}_S \mathbb{P}^8$. The task is to understand this blow-up, for example by constructing a smooth variety V dominating it.

Goal: $s(S, \mathbb{P}^8)$.

The problem of computing Segre classes in projective space is difficult and important.

Definition

Let $q = (q_0 : q_1 : q_2) \in \mathbb{P}^2$. The **point condition** determined by q is the hypersurface X_q of \mathbb{P}^8 defined by $F(\varphi(q_0 : q_1 : q_2)) = 0$.

The base scheme of α is $S = \bigcap_{q \in \mathbb{P}^2} X_q$.

Set-theoretically: $\varphi \in S \iff \text{im } \varphi \subseteq C$:

- S has one irreducible component for each component of C ;
- Linear components of $C \rightsquigarrow$ 5-dimensional components of S ;
- Nonlinear components of $C \rightsquigarrow$ 3-dimensional components.

$L \subseteq C$ line, wlog $x_0 = 0$.

Corresponding component of S : $\varphi \in \mathbb{P}^8$ s.t. $\text{im } \varphi \subseteq L$.

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \in L \text{ for all } x_0, x_1, x_2$$

$$\iff a_{00}x_0 + a_{01}x_1 + a_{02}x_2 = 0 \text{ for all } x_0, x_1, x_2$$

$$\iff a_{00} = a_{01} = a_{02} = 0: \text{ a linear } \mathbb{P}^5 \subseteq \mathbb{P}^8.$$

$C =$ line arrangement in $\mathbb{P}^2 \rightsquigarrow S =$ arrangement of \mathbb{P}^5 's in \mathbb{P}^8 .

$L_1 \neq L_2 \rightsquigarrow$ corresponding \mathbb{P}^5 's meet along a \mathbb{P}^2 .

This case is in many ways similar to the case of d -tuples of points in \mathbb{P}^1 .

The other end of the scale:

C = irreducible curve of degree $d > 1$.

Base locus: irreducible case

The base scheme of α is supported on $\mathbb{P}^2 \times C \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \subseteq \mathbb{P}^8$.

$\mathbb{P}^2 \times \mathbb{P}^2$: rank-1, 3×3 matrices.

Factors corresponding to kernel line and image point.

Along S , the kernel is free but the image is constrained to be a point of C .

First: Understand the situation at smooth points of C .

In fact, **assume C is smooth**.

$S_{\text{supp}} = \mathbb{P}^2 \times C$: smooth *support*. Is S smooth?

No! Interesting nilpotent structure.

That's what makes this problem so challenging.

THE OPTIMIST'S STRATEGY:

W : nonsingular variety; $S \subseteq W$: subscheme;
assume $B := S_{\text{supp}}$ is nonsingular.

Want: Proper birational $\widetilde{W} \rightarrow W$ dominating $B\ell_S W$.

- $S = \cap_i X_i$;
- $W^1 :=$ blow-up of W along $B = S_{\text{supp}}$;
- $X_i^1 :=$ proper transform of X_i ;
- $S^1 := \cap_i X_i^1$. If $S^1 = \emptyset$, done!
- If not, hope that S_{supp}^1 is nonsingular; (we are optimists!)
- Repeat.
- $\widetilde{W} = W^r$ for $r \gg 0$. (Again, we are optimists.)

Theorem

For *smooth* C , the optimist's strategy works, with $i =$ maximum order of contact of C with a line.

- Begin with $S \subseteq V^0 := \mathbb{P}^8$; $S = \cap_{q \in \mathbb{P}^2} X_q$;
- $B := S_{\text{supp}} = \mathbb{P}^2 \times C$ is nonsingular, $\dim = 3$;
- $V^1 := B\ell_B V^0$; $X_q^1 :=$ proper transform of X_q ; $S^1 := \cap_q X_q^1$;
- $B^1 := S_{\text{supp}}^1$ is nonsingular! In fact, a \mathbb{P}^1 -bundle over B ;
- $V^2 := B\ell_{B^1} V^1$; $X_q^2 :=$ proper transform of X_q^1 ; $S^2 := \cap_q X_q^2$;
- $B^2 := S_{\text{supp}}^2$ is nonsingular! In fact, S_{supp}^2 is a union of smooth 3-folds, one for each flex of C ;
- $V^3 := B\ell_{B^2} V^2$; $X_q^3 :=$ proper transform of X_q^2 ; $S^3 := \cap_q X_q^3$;
- $B^3 := S_{\text{supp}}^3$ is nonsingular! In fact, B^3 is a union of smooth 4-folds, one for each hyperflex of C .

Note $B^3 = \emptyset$ if C only has ordinary flexes. Thus, 3 blow-ups suffice in this case.

Past this stage, the blow-ups admit a uniform description: for $i \geq 4$,

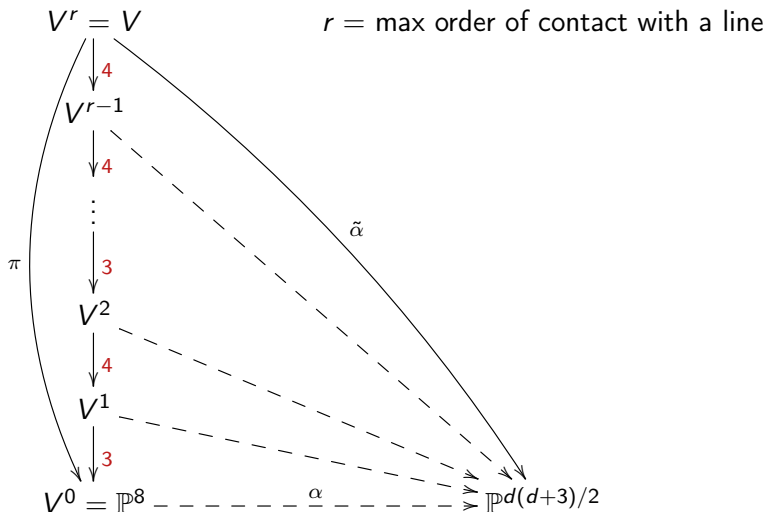
- $V^i := \text{Bl}_{B^{i-1}} V^{i-1}$; $X_q^i :=$ proper transf. of X_q^{i-1} ; $S^i := \cap_q X_q^i$;
- $B^i := S_{\text{supp}}^i$ is nonsingular! In fact, B^i is a union of smooth 4-folds, one for each point of C at which the tangent line meets C with intersection multiplicity $> i$ at that point.

So if the max order of contact of a line with C is r , then $S^r = \emptyset$, construction stops at that stage.

I.e., V^r dominates the closure of the graph of α ,

i.e., $V = V^r$ resolves the indeterminacies of α .

Basic diagram



Further details? (Nice geometry!)

What is B^1 ?

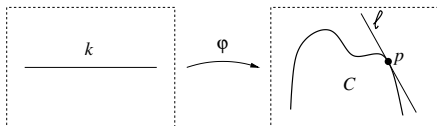
$E^1 :=$ exceptional divisor in V^1 . Then $E^1 \cong \mathbb{P}(N_{\mathbb{P}^2 \times C} \mathbb{P}^8)$.

By construction, $\cap_{q \in \mathbb{P}^2} X_q^1 \subseteq E^1$.

Therefore, analyze situation over $\varphi \in \mathbb{P}^2 \times C$.

$\text{rk } \varphi = 1$: φ is determined by **kernel line k** and **image point $p \in C$** .

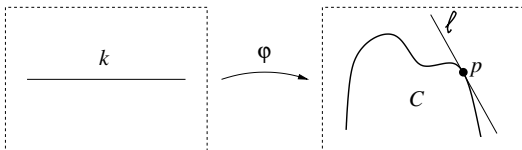
C is smooth at p by assumption: **$\ell =$ tangent line to C at p** .



Lemma

The tangent space to $\mathbb{P}^2 \times C$ at φ consists of all $\psi \in \mathbb{P}^8$ such that $\text{im } \psi \subseteq \ell$ and $\psi(k) \subseteq p$.

(E.g., $\text{im } \varphi = p \in \ell$, $\varphi(k) = \emptyset \subseteq p$.)



Lemma

The tangent space to $\mathbb{P}^2 \times C$ at φ consists of all $\psi \in \mathbb{P}^8$ such that $\text{im } \psi \subseteq \ell$ and $\psi(k) \subseteq p$.

Proof:

Both spaces are linear subspaces of \mathbb{P}^8 of dimension 3 and contain spanning subspaces $\mathbb{P}^2 = \{(*, p)\}$ and $\mathbb{P}^1 = \{(k, q) \mid q \in \ell\}$. \square

What about X_q ?

Lemma

For all $q \in \mathbb{P}^2$, X_q is nonsingular at φ . The tangent space to X_q at φ consists of all $\psi \in \mathbb{P}^8$ such that $\psi(q) \subseteq \ell$.

Lemma

For all $q \in \mathbb{P}^2$, X_q is nonsingular at φ . The tangent space to X_q at φ consists of all $\psi \in \mathbb{P}^8$ such that $\psi(q) \subseteq \ell$.

Proof:

Equation for X_q : $F(\psi(q)) = 0$. Restrict to line $\varphi + t\psi$, expand:

$$F(\varphi(q)) + \sum_i \left(\frac{\partial F}{\partial x_i} \right)_{\varphi(p)} \psi_i(q) t + \cdots = 0$$

(where $\psi_i(q)$ denotes the i -th coordinate of $\psi(q)$).

$F(\varphi(q)) = 0$ since $\text{im } \varphi = q \in C$.

ψ in tangent space \iff linear term vanishes $\iff \psi(q) \subseteq \ell$.

If $\psi(q) \not\subseteq \ell$, get int. mult. = 1, $\implies X_q$ nonsingular at φ . □

Lemma

The tangent space to $\mathbb{P}^2 \times C$ at φ consists of all $\psi \in \mathbb{P}^8$ such that $\text{im } \psi \subseteq \ell$ and $\psi(k) \subseteq p$.

Lemma

For all $q \in \mathbb{P}^2$, X_q is nonsingular at φ . The tangent space to X_q at φ consists of all $\psi \in \mathbb{P}^8$ such that $\psi(q) \subseteq \ell$.

Therefore:

$$\begin{aligned} T_{\varphi} B &= \{\psi \in \mathbb{P}^8 \mid \text{im } \psi \subseteq \ell, \psi(k) \subseteq p\} \cong \mathbb{P}^3, \\ \cap_q T_{\varphi} X_q &= \{\psi \in \mathbb{P}^8 \mid \text{im } \psi \subseteq \ell\} \cong \mathbb{P}^5. \end{aligned}$$

This shows that $\dim(\cap_q T_{\varphi} X_q)/(T_{\varphi} B) = 2$, hence (set-th.):

$$\cap_q X_q^1 = \cap_q (X_q^1 \cap E^1) = \mathbb{P}(\cap_q (TX_q/TB))$$

is a \mathbb{P}^1 bundle over $B = \mathbb{P}^2 \times C$. **This is B^1 .**

Contribution of inflection points

Where do inflection points come in?

$$S^1 := \cap_q X_q^1; B^1 := S_{\text{supp}}^1 = \cap_q (X_q^1 \cap E^1)$$

Note: $B^1 = S^1 \cap E^1$ (scheme-theoretically).

Consequence: In $V^2 = \text{Bl}_{B^1} V^1$, $(\cap_q X_q^2) \cap \tilde{E}^1 = \emptyset$.

Consequence: In $V^2 = \text{Bl}_{B^1} V^1$, $S^2 := (\cap_q X_q^2)$ consists of ≤ 1 point over every point of B^1 .

Proof: Fibers of E^2 are projective spaces. Fibers of S^2 are linear subspaces, and they miss the hyperplane $\tilde{E}^1 \cap E^2$. □

This shows that $B^2 := S_{\text{supp}}^2$ consists of a section of E^2 over a subset of B^1 . Which subset?

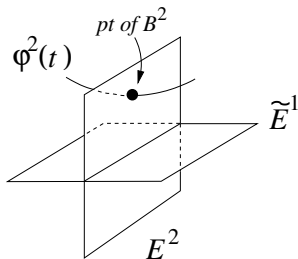
Claim

The inverse image of $\mathbb{P}^2 \times \{p\}$ with p an *inflection point* of C .

Lemma

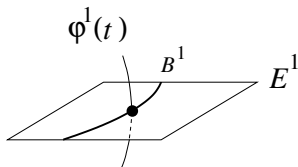
B^2 is the union of the sections of B^1 over $\mathbb{P}^2 \times \{p\}$, p an inflection point of C .

Proof? Key tool here: In V^2 ...



Consider a small arc $\varphi^2(t)$ centered at a point of B^2 , and transversal to E^2 . The order of vanishing of S^2 along $\varphi^2(t)$ is ≥ 1 . Note that we know $\varphi^2(t)$ may be chosen to be disjoint from \tilde{E}^1 . Push $\varphi^2(t)$ down to $\varphi^1(t)$ in V^1 .

In $V^1 \dots$

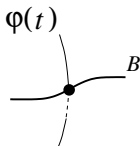


The push-forward $\varphi^1(t)$ is a germ centered at a point of B^1 .
 Important: $\varphi^1(t)$ is nonsingular and transversal to E^1
 (because $\varphi^2(t)$ missed \tilde{E}^1).

The order of vanishing of S^1 along $\varphi^1(t)$ is ≥ 2
 (because it is $\geq 1 +$ order of vanishing of E^2 along $\varphi^2(t)$).

Push $\varphi^1(t)$ down to $\varphi(t)$ in $V^0 = \mathbb{P}^8$.

In $\mathbb{P}^8 \dots$



The push-forward $\varphi(t)$ is a germ centered at a point of B .
 $\varphi(t)$ is nonsingular and normal to B
(because $\varphi^1(t)$ was transversal to E_1).

The order of vanishing of S along $\varphi(t)$ is ≥ 3
(because it is $\geq 1 +$ order of vanishing of E^1 along $\varphi^1(t)$).

Conclusion

Points of B^2 correspond to nonsingular germs of curve $\varphi(t)$

- centered at points of B and normal to it, and
- such that S vanishes to order ≥ 3 along $\varphi(t)$.

Definition

S : subscheme of a nonsingular variety; B : support of S .

The **thickness** of S at $p \in B$ is the maximum order of vanishing of S along a nonsingular germ of curve centered at p and normal to B .

Recall that the aim was to show:

Lemma

B^2 is the union of the sections of B^1 over $\mathbb{P}^2 \times \{p\}$, p an inflection point of C .

We have verified that B^2 dominates the locus in $B = \mathbb{P}^2 \times C$ where the thickness of S is ≥ 3 , and B^2 has no components over points where the thickness of S is ≤ 2 .

So we are reduced to a **thickness** computation.

Lemma

Let $\varphi \in B$, with $\text{im } \varphi = p \in C$. Then

$$\text{th}_\varphi(S) = \text{order of contact of } C \text{ with its tangent line at } p.$$

Proof:

By definition, $\text{th}_\varphi(S)$ is the maximum order of contact of a nonsingular germ $\varphi(t)$ normal to B and s.t. $\varphi(0) = \varphi$ with a point-condition X_q .

Let m be the order of contact of C with its tang. line at $p = \text{im } \varphi$. To show $\text{th}_\varphi(S) \geq m$, enough to produce $\varphi(t)$ s.t. order of contact with X_q is $\geq m$.

$\varphi(t) := \varphi + \psi t$ such that

- $\text{im } \psi = \mathbb{T}_p C$
- $\psi(\ker \varphi) \neq p$.

Then $\varphi(t)$ is normal to B (second condition), and $\varphi(t)(q)$ parametrizes $\mathbb{T}_p(C)$.

$\varphi(t)(q)$ parametrizes $\mathbb{T}_p(C)$

Then $X_q \cdot \varphi(t) =$ order of vanishing of $F(\varphi(t)(q))$
= order of vanishing of F along $\mathbb{T}_p C$
= order of contact of C with $\mathbb{T}_p C$ at $p = m$.

So $\text{th}_\varphi(C) \geq m$.

$\text{th}_\varphi(C) \leq m$: analogous computation. □

This concludes the sketch of the proof of our description of B_2 :

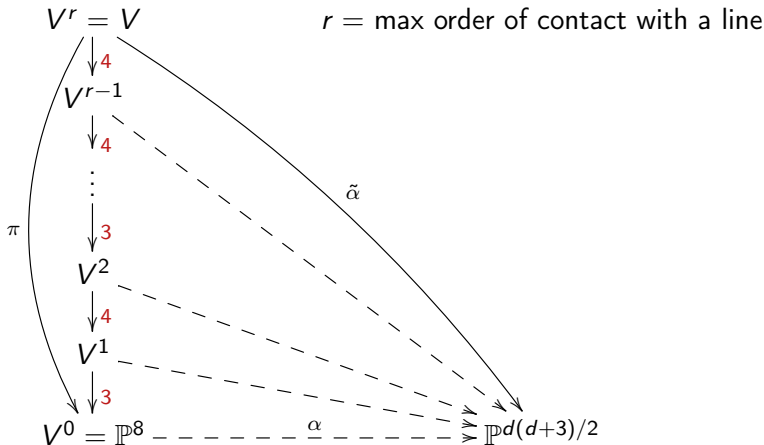
Lemma

B^2 consists of a union of \mathbb{P}^1 -bundles over $\mathbb{P}^2 \times \{p\}$, p an inflection point of C .

The same technique is used to study further blow-ups.

Summary

The base locus of the lift $V_i \dashrightarrow \mathbb{P}^{d(d+3)/2}$ lives over $\mathbb{P}^2 \times \{p\}$, where $p \in C$ are points such that $(C \cdot \mathbb{T}_p C)_p > i$.



What was this all about?

Here is the basic diagram again:

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^{d(d+3)/2}
 \end{array}
 \quad \varphi \in \mathbb{P}^8 \mapsto \alpha(\varphi) := C \circ \varphi$$

$$\overline{O_C} = \overline{\mathrm{im} \alpha}.$$

At this point, we have explicitly constructed a V filling this diagram, with a proper birational map to \mathbb{P}^8 , **provided C is smooth**.

We have discovered that the construction only depends on the number and type of inflection points of C .

For example: If C only has ordinary flexes, then V may be realized by a 3-stage blow-up at smooth centers over \mathbb{P}^8 .

The construction is now ready to be used to study $\overline{O_C}$.

The Segre class

Reminder: The enumerative information is captured by the **predegree polynomial** of C , and that information is encoded in the **Segre class** of the base scheme S of α :

Theorem

Let $\iota : S \hookrightarrow \mathbb{P}^8$ be the base scheme of the rational map α associated with a curve C . Then the predegree polynomial of C equals

$$\frac{([\mathbb{P}^8] - \iota_* s(S, \mathbb{P}^8)) \otimes \mathcal{O}(-dH)}{1 - dH}$$

(See the d -tuple story.)

$s(S, \mathbb{P}^8)$ can be extracted from $\pi : V \rightarrow \mathbb{P}^8$ in essentially the same way used in the (much simpler) case of d -tuples.

The end-result is

Theorem

Let C be a smooth curve of degree d , and let S be the base scheme of the action map $\mathbb{P}^8 \dashrightarrow \mathbb{P}^{d(d+3)/2}$. Then $s(S, \mathbb{P}^8)$ equals

$$\begin{aligned} &12dH^5 + d(25d - 162)H^6 - 48d(9d - 31)H^7 + 3d(1325d - 3546)H^8 \\ &\quad + \sum_{p \in C} (\nu - 2)(\nu - 3) \cdot \\ &\quad \cdot ((\nu + 5)H^6 - 3(\nu^2 + 6\nu + 24)H^7 + 3(2\nu^3 + 13\nu^2 + 55\nu + 197)H^8) \end{aligned}$$

where ν = order of contact of C and $\mathbb{T}_p C$.

Note $\nu = 2$ for all but fin. many points of C , so the $\sum_{p \in C}$ is finite.
(This can be carried out in positive characteristic, but the blow-ups must be modified if e.g., every point of C is an inflection point.)

In fact, the $\sum_{p \in C} = 0$ if all inflection points of C are ordinary.

Example

Assume C only has ordinary flexes. Then $s(S, \mathbb{P}^8)$ equals

$$12dH^5 + d(25d - 162)H^6 - 48d(9d - 31)H^7 + 3d(1325d - 3546)H^8$$

For $d \geq 3$ ($\dim O_C = 8$) this corresponds to a predegree of

$$\begin{aligned} d^8 - 1372d^4 + 7992d^3 - 15879d^2 + 10638d \\ = d(d-2)(d^6 + 2d^5 + 4d^4 + 8d^3 - 1356d^2 + 5280d - 5319) \end{aligned}$$

The fact that this vanishes for $d = 2$ signals that the orbit closure of a smooth conic has dimension < 8 . For $d = 2$,

$$s(S, \mathbb{P}^8) = 24H^5 - 224H^6 + 1248H^7 - 5376H^8$$

$$s(S, \mathbb{P}^8) = 24H^5 - 224H^6 + 1248H^7 - 5376H^8$$

Predegree polynomial:

$$\begin{aligned} & \frac{(1 - 24H^5 + 224H^6 - 1248H^7 + 5376H^8) \otimes \mathcal{O}(-2H)}{1 - 2H} = \frac{1}{1 - 2H} \\ & - 24 \frac{H^5}{(1 - 2H)^6} + 224 \frac{H^6}{(1 - 2H)^7} - 1248 \frac{H^7}{(1 - 2H)^8} + 5376 \frac{H^8}{(1 - 2H)^9} \\ & = 1 + 2H + 4H^2 + 8H^3 + 16H^4 + \boxed{8}H^5 \quad . \end{aligned}$$

Of course $\overline{\mathcal{O}_C} = \mathbb{P}^5$ for a smooth conic!

This is a complicated way to compute the degree ($= 8$) of the $\mathrm{PGL}(3)$ stabilizer of a smooth conic.

Remarks

- The Segre class (hence the predegree polynomial) for a smooth curve depends only on the degree of C and the number and order of flexes, not on their position or on other features of the curve.
- In fact, the predegree is determined by d and $f\ell_2, \dots, f\ell_5$, where $f\ell_i := \sum_{p \in C} (\nu - 2)^i$.
- The contribution of a special point to the Segre class is *independent of d* . For example, the contribution of a 'hyperflex' ($\nu = 4$) is $6H^6(3 - 64H + 753H^2)$.
- The contribution to the predegree depends on d . For example, a hyperflex contributes $(-504d^2 + 3072d - 4518)$.

Example

Degree of trisecant variety to d -th Veronese of \mathbb{P}^2 ?

This is the orbit closure of the Fermat curve $x^d + y^d + z^d$, a curve of degree d with $3d$ flexes with $\nu = d$.

Segre class $= 12dH^5 + d(3d^3 - 32d - 72)H^6 - 3d(3d^4 + 3d^3 - 108d - 64)H^7 + 3d^2(6d^4 + 9d^3 + 6d^2 - 640)H^8$

\rightsquigarrow predegree $d^2(d-2)(d^5 + 2d^4 - 26d^3 - 7d^2 + 192d - 192)$.

Order of stabilizer: $6d^2$. Therefore, the degree of the trisecant variety to the d -th Veronese ($d \geq 3$) is

$$\frac{(d-2)(d^5 + 2d^4 - 26d^3 - 7d^2 + 192d - 192)}{6}.$$

SMALL ORBITS

$C \subseteq \mathbb{P}^2$: curve, degree d :

$C = V(F)$, $F(x_0, x_1, x_2)$ homogeneous, degree d .

$\mathrm{PGL}(3)$ acts on C :

$\varphi \in \mathrm{PGL}(3) \rightsquigarrow C \circ \varphi := F(\varphi(x_0, x_1, x_2)) \in \mathbb{P}^{d(d+3)/2}$.

$O_C := \text{orbit}$, $\overline{O_C} := \text{orbit closure}$. Main focus: $\deg \overline{O_C}$.

Basic diagram: $\overline{O_C} = \overline{\mathrm{im} \alpha}$,

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^{d(d+3)/2}
 \end{array}
 \quad \varphi \in \mathbb{P}^8 \mapsto \alpha(\varphi) := C \circ \varphi$$

Recall: For *smooth* curves C , we were able to construct a suitable V explicitly by a sequence of blow-ups at smooth centers.

The “Optimist’s Strategy” works in this case.

Next stop: Curves with **small orbit**, i.e., C s.t. $\dim O_C < 8$.

How special is for a curve to have small orbit? Very.

In fact, the classification of such curves is rather compact:

Theorem

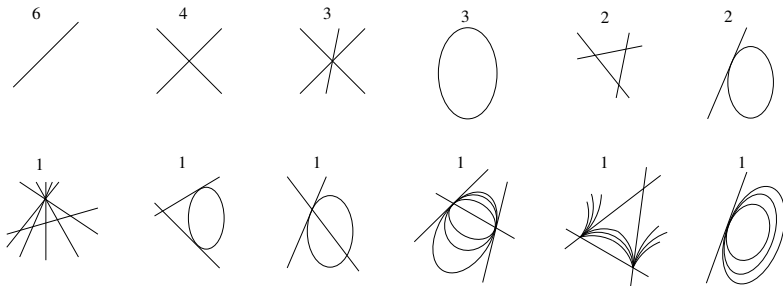
Let C be a curve such that $\dim O_C < 8$. Then, up to $\mathrm{PGL}(3)$ -translations

- *C_{supp} is a union of components from the collection of lines $x = 0, y = 0, z = 0$ and irreducible curves $y^b + \lambda z^a x^{b-a}$ (for fixed $0 \leq 2a \leq b$); or*
- *C_{supp} is a union of components from the collection of curves $\lambda x^2 + \alpha xy + \beta xz + \gamma y^2$ (for fixed α, β, γ) and the line $x = 0$.*

A picture is worth a thousand words...

Curves with small orbits

Representative pictures of curves with small orbit, including dimension of stabilizer:



All but the last one correspond to curves from the first item in the theorem; the last picture correspond to the second item.

In the last picture, the conics are **quadritangent**: they meet exactly at one point.

Proof of the theorem?

If the stabilizer has positive dimension, then it must contain a 1-dimensional subgroup.

Only two possibilities: \mathbb{G}_m or \mathbb{G}_a .

- \mathbb{G}_m : May be diagonalized, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & t^a & 0 \\ 0 & 0 & t^b \end{pmatrix}$, $0 \leq 2a \leq b$.

Irreducible fixed curves: $x = 0$, $y = 0$, $z = 0$, $y^b + \lambda z^a x^{b-a}$.

- \mathbb{G}_a : May be put in standard form $\begin{pmatrix} 1 & 0 & 0 \\ at & 1 & 0 \\ bt + \frac{1}{2}act^2 & ct & 1 \end{pmatrix}$.

Again, can list fixed curves \rightsquigarrow last item in the classification.



Task: To construct a variety V resolving the rational map extending the action.

Basic diagram: $\overline{O_C} = \overline{\text{im } \alpha}$,

$$\begin{array}{c}
 V \\
 \swarrow \pi \quad \searrow \tilde{\alpha} \\
 \text{PGL}(3) \hookrightarrow \mathbb{P}^8 \xrightarrow{\quad \alpha \quad} \mathbb{P}^{d(d+3)/2}
 \end{array}
 \quad \varphi \in \mathbb{P}^8 \mapsto \alpha(\varphi) := C \circ \varphi$$

Recall: The base locus of α is supported on $\{\varphi \in \mathbb{P}^8 \mid \text{im } \varphi \subseteq C\}$. For example, if C has no linear components, then the base locus is $\mathbb{P}^2 \times C$ (and expect the base **scheme** to have interesting nilpotents).

Can we apply the “Optimist’s Strategy” here?

No: The base locus is singular in most cases.

OPTIMIST'S STRATEGY, take 2

In the smooth case, V is obtained by 2 'global' blow-ups and several 'local' ones:

- Blow up along $B = \mathbb{P}^2 \times C$;
- Blow up along a \mathbb{P}^1 -bundle B_1 over B ;
- Blow up many loci lying over $\mathbb{P}^2 \times \{p\}$, p flexes on C .

Moral: There are special points (inflection points on a smooth C), and the base *scheme* is extra thick on corresponding loci.

Alternative: Blow-up along these special loci first, then deal with the global centers.

Claim

For C smooth, this works!

Does it work for singular curves?

Revised Optimist's Strategy

- Identify special points of C , e.g., inflection points, singularities;
- Blow-up along corresponding loci in the base locus B , possibly several times;
- **Hope** that after suitable blow-ups, special points no longer look special (we are still fairly optimistic!);
- Complete the process by blowing up twice.

The case of C smooth can be dealt with along these lines, producing a different V (but computing the same Segre class).

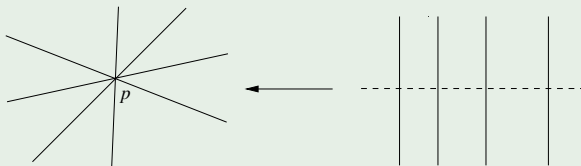
For singular curves, e.g., most curves with small orbits, we have no choice: we have to adopt this Revised Optimist's Strategy.

Idea: Since we have to 'fix' the special points of the curve, try to mirror an **embedded resolution** of C (as well as taking care of flexes).

Example

If C is a union of **lines**, this is particularly straightforward.

- To obtain an embedded resolution, simply blow-up the vertices of the configuration.



- Correspondingly, blow-up \mathbb{P}^8 along $\bigcup_p \text{vertex } \mathbb{P}^2 \times \{p\}$;
- Then blow-up along proper transform of $\bigcup_{\ell} \text{line } \mathbb{P}_{\ell}^5$, where $\mathbb{P}_{\ell}^5 := \{\varphi \mid \text{im } \varphi \subseteq \ell\}$.
- This resolves $\alpha \rightsquigarrow$ computation of the Segre class.

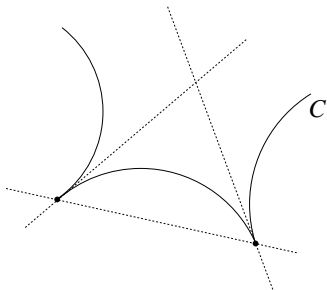
So the 'revised Optimist's Strategy' takes care of 5 of the 12 different types of curves with small orbit.

(In fact, it works for **all** line configurations, small or large.)

Theorem

The revised Optimist's Strategy resolves the basic rational map for all curves with small orbit.

Typical singular irreducible component of curve with small orbit ('type I'): $x^n = y^m z^{n-m}$, $0 \leq m \leq n$, m, n relatively prime.



Singularity at $(0 : 0 : 1)$: $x^n = y^m$. Embedded resolution \leftrightarrow Euclidean algorithm for (m, n) .

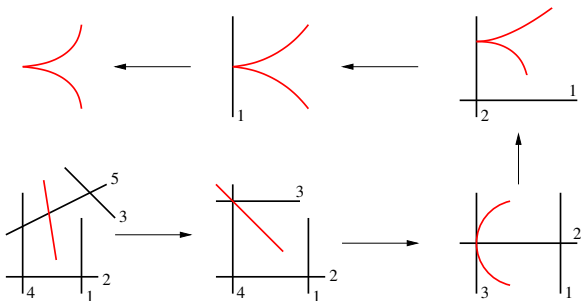
Example ($x^8 = y^3$)

$$8 = 3 \cdot 2 + 2$$

$$3 = 2 \cdot 1 + 1$$

$$2 = 1 \cdot 2$$

\leadsto need $2 + 1 + 2 = 5$ blow-ups to resolve the singularity.



It is useful to give a more compact description of this process.

$B \subseteq P \subseteq V$ nonsingular varieties.

$V^{(1)} := B\ell_B V$; $E^{(1)}$ = exceptional divisor.

For $j > 1$, let $V^{(j)} := B\ell_{\tilde{P} \cap E^{(j-1)}} V^{(j-1)}$.

Definition

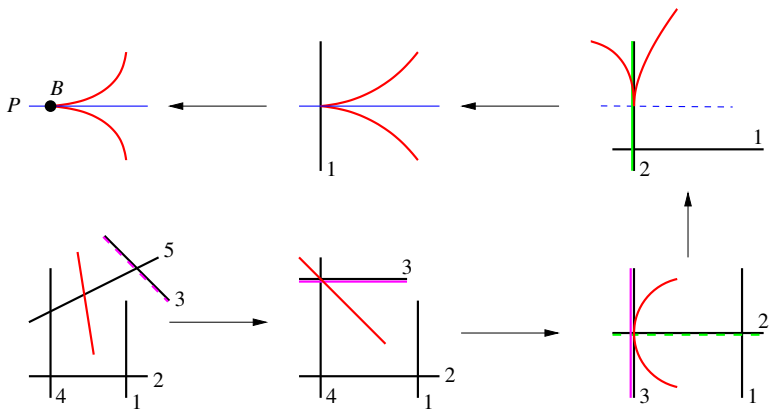
$V^{(\ell)}$ is the ℓ -th directed blow-up of V along B in the direction of P .

Each line of the Euclidean algorithm,

$$m_{i-2} = m_{i-1} \cdot \ell_i + m_i$$

corresponds to an ℓ_i -directed blow-up.

For example, $x^8 = y^3$ is resolved by 3 directed blow-ups (2-, 1-, 2-).



Recipe to resolve $\alpha : \mathbb{P}^8 \dashrightarrow \mathbb{P}^{d(d+3)/2}$ for $C : x^n = y^m$, near $\mathbb{P}^2 \times \{p\}$ ($p = \text{singular point}$). Euclidean algorithm:

$$n = m \cdot \ell_1 + m_1$$

$$m = m_1 \cdot \ell_2 + m_2$$

$$m_1 = m_2 \cdot \ell_3 + m_3$$

...

- ℓ_1 -directed blow-up of \mathbb{P}^8 along $B = \mathbb{P}^2 \times \{p\}$ [$\dim = 2$] in the direction of $P = \mathbb{P}^5 = \{\varphi \mid \text{im } \varphi \subseteq \mathbb{T}_p C\}$.
 $E_1 := \text{last exceptional divisor}$, $\tilde{P} := \text{last proper transform of } P$.
- ℓ_2 -directed blow-up along $\tilde{P} \cap E_1$ [$\dim = 4$] in the dir. of E_1 .
- ℓ_3 -directed blow-up along $\tilde{E}_1 \cap E_2$ [$\dim = 6$] in the dir. of E_2 .
- etc.
- At the end of this process, two ‘global’ blow-ups suffice to resolve α (near $\mathbb{P}^2 \times \{p\}$).

So this procedure equates p to ordinary nonsingular points of C in terms of their contribution to the base scheme of α .

Further needed work:

- Must allow for several components, with multiplicities s_i .
- Take care of possible linear components (on the triangle).
- Treat 'quadritangent conics'. ('Here a miracle happens. . .')
- Do the intersection theory!

Very messy induction.

Good news: Massive simplifications, the Segre class depends directly on the exponents, not on individual steps of the Euclidean algorithm. So, the following should be seen as 'simple':

- Expand the expression

$$\begin{aligned} n^2 m^2 \bar{m}^2 \left(\left(s + \frac{r}{n} + \frac{q}{m} + \frac{\bar{q}}{\bar{m}} \right)^7 + 2 \left(s + \frac{r}{n} + \frac{q}{m} \right)^7 + 2 \left(s + \frac{r}{n} + \frac{\bar{q}}{\bar{m}} \right)^7 \right. \\ \left. + \left(s + \frac{r}{n} \right)^7 - 42 \left(s + \frac{r}{n} \right)^5 \left(\frac{q^2}{m^2} - \frac{q}{m} \frac{\bar{q}}{\bar{m}} + \frac{\bar{q}^2}{\bar{m}^2} \right) \right) \end{aligned}$$

and set $r^i = q^i = \bar{q}^i = 0$ for $i \geq 3$. Get a *polynomial*.

- Subtract

$$(84(Sn + r + q + \bar{q})^2 \sum s_i^5 - 252(Sn + r + q + \bar{q}) \sum s_i^6 + 192 \sum s_i^7)$$

and set $S = \sum s_i$.

- Get a polynomial expression $Q(n, m, s_i, r, q, \bar{q})$.

Theorem

If 7-dimensional, the orbit closure of a curve consisting of curves of type $x^n = y^m z^{\bar{m}}$ ($m + \bar{m} = n$), appearing with multiplicity s_i , and lines from the frame triangle, with multiplicities r, q, \bar{q} , has degree

$$\deg \overline{O_C} = \frac{1}{A} \cdot Q(n, m, s_i, r, q, \bar{q})$$

where A is the number of components of $\text{Stab}(C)$.

This comes from the coefficient of H^7 in the a.p.p.

If $\dim O_C < 7$, then the expression equals 0.

Example

The orbit closure of $x^n = y^m z^{n-m}$ has a.p.p.

$$\begin{aligned} 1 + nH + \frac{n^2 H^2}{2!} + \frac{n^3 H^3}{3!} + \frac{n^4 H^4}{4!} + \frac{n(n^4 - 12)H^5}{5!} \\ + \frac{3n(n^3 m(n - m) - 16n + 24)H^6}{6!} \\ + \frac{6n(n^2 m^2(n - m)^2 - 14n^2 + 42n - 32)H^7}{7!} \end{aligned}$$

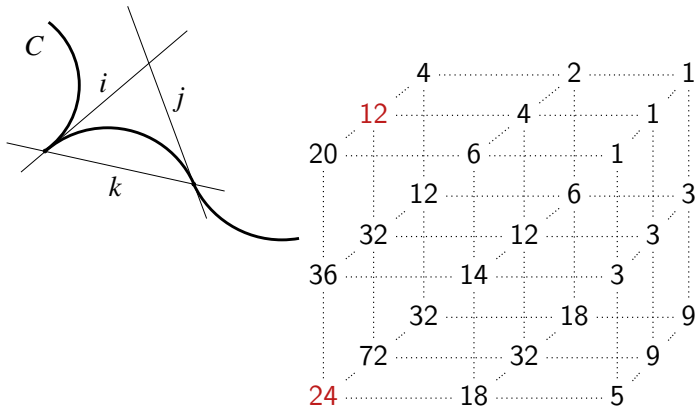
E.g., $x^2 = yz$: $1 + 2H + \frac{4H^2}{2!} + \frac{8H^3}{3!} + \frac{16H^4}{4!} + \frac{8H^5}{5!}$, agreeing with previous computation for smooth conic.

$x^3 = y^2 z$ (i.e., cuspidal cubic):

$$1 + 3H + \frac{9H^2}{2!} + \frac{27H^3}{3!} + \frac{81H^4}{4!} + \frac{207H^5}{5!} + \frac{270H^6}{6!} + \frac{72H^7}{7!}$$

(so $\deg = 72/3 = 24$, as it should).

Remark: Standard enumerative information can be extracted from the a.p.p. E.g., get number of curves with constraints on the lines of the frame triangle. For cuspidal cubics, these numbers were known to Schubert. (Modern work of Miret-Xambo, **D. Nguyen.**)



E.g.: $(i, j, k) = (0, 0, 0) \rightsquigarrow 24$, $(i, j, k) = (0, 1, 2) \rightsquigarrow 12$.

ARBITRARY
CURVES
LIMITS
OF PGL
TRANS
LATES

$C \subseteq \mathbb{P}^2$: curve, degree d :

$C = V(F)$, $F(x_0, x_1, x_2)$ homogeneous, degree d .

$\mathrm{PGL}(3)$ acts on C :

$\varphi \in \mathrm{PGL}(3) \rightsquigarrow C \circ \varphi := F(\varphi(x_0, x_1, x_2)) \in \mathbb{P}^{d(d+3)/2}$.

$O_C := \text{orbit}$, $\overline{O_C} := \text{orbit closure}$. Main focus: $\deg \overline{O_C}$.

Basic diagram: $\overline{O_C} = \overline{\mathrm{im} \alpha}$,

$$\begin{array}{ccc}
 & V & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 \mathrm{PGL}(3) \hookrightarrow \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^{d(d+3)/2}
 \end{array}
 \quad \varphi \in \mathbb{P}^8 \mapsto \alpha(\varphi) := C \circ \varphi$$

Recall: For *smooth* curves C , and for curves *with small orbit*, we were able to construct a suitable V explicitly by a sequence of blow-ups at smooth centers.

Various “Optimist’s Strategies” work in these cases.

Arbitrary curves

Problem: These strategies are too messy for an arbitrary curve. (They may work for what I know, but we have not been able to carry them out in complete generality.)

Alternative?

Idea: Work directly with $\bar{\Gamma} :=$ the closure of the graph of the rational map α .

$$\begin{array}{ccc} & \boxed{\bar{\Gamma}} & \\ \pi \swarrow & & \searrow \tilde{\alpha} \\ \mathbb{P}^8 & \xrightarrow{\alpha} & \mathbb{P}^{d(d+3)/2} \end{array} \quad \bar{\Gamma} \subseteq \mathbb{P}^8 \times \mathbb{P}^{d(d+3)/2}$$

Clear: The (adjusted) predegree polynomial may be recovered from the class of $\bar{\Gamma}$ in $\mathbb{P}^8 \times \mathbb{P}^{d(d+3)/2}$.

Clear modulo intersection theory: It may also be recovered from $\pi^{-1}(S)$, where S is the base *scheme* of α .

Clear modulo intersection theory: The predegree polynomial may be recovered from $\pi^{-1}(S)$, where S is the base *scheme* of α . Indeed, recall:

Theorem

Let $\iota : S \hookrightarrow \mathbb{P}^8$ be the base scheme of the rational map α associated with a curve C . Then the predegree polynomial of C equals

$$\frac{([\mathbb{P}^8] - \iota_* s(S, \mathbb{P}^8)) \otimes \mathcal{O}(-dH)}{1 - dH}$$

and

Lemma (Birational invariance of Segre classes)

$$s(S, \mathbb{P}^8) = \pi_* s(\pi^{-1}(S), \bar{\Gamma})$$

So it is clear the information is captured by $\pi^{-1}(S)$.

In fact, it is captured by the class $[\pi^{-1}(S)]$ in $\mathbb{P}^8 \times \mathbb{P}^N$. Why?
 $(N = d(d+3)/2)$

Notation:

- h : hyperplane class in \mathbb{P}^N
- H : hyperplane class in $W = \mathbb{P}^8$.
- $\mathcal{L} := \mathcal{O}(dH)$; $\mathbb{P}^N = \mathbb{P}(\mathcal{E}^\vee)$, $\mathcal{E} \subseteq H^0(\mathbb{P}^8, \mathcal{L})$. $\ell := c_1(\mathcal{O}(\mathcal{L}))$.
- $\widetilde{\mathcal{L}} := \tilde{\alpha}^* \mathcal{O}(h)$; (shorthand: $h = c_1(\widetilde{\mathcal{L}})$ on $\bar{\Gamma}$)
- a.p.p. : $\pi_*(ch(\widetilde{\mathcal{L}}) \cap [\bar{\Gamma}]) = 1 + a_1 H + a_2 \frac{H^2}{2!} + a_3 \frac{H^3}{3!} + \dots$
- $S :=$ base scheme of $\alpha =$ scheme defined by all sections in \mathcal{E} .

$$\begin{array}{ccc}
 & \bar{\Gamma} & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 W & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

$$S = \emptyset \iff \alpha \text{ regular.}$$

$$\begin{array}{ccc}
 & \bar{\Gamma} & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 W & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

- \mathcal{L} : line bundle on W ; $\mathbb{P}^N = \mathbb{P}(\mathcal{E}^\vee)$, $\mathcal{E} \subseteq H^0(W, \mathcal{L})$
- $\widetilde{\mathcal{L}} = \tilde{\alpha}^* \mathcal{O}(1)$.
- $\ell := c_1(\mathcal{O}(\mathcal{L}))$ (shorthand: $\ell = c_1(\pi^* \mathcal{L})$); $h := c_1(\mathcal{O}(\widetilde{\mathcal{L}}))$.
- a.p.p. : $\pi_*(ch(\widetilde{\mathcal{L}}) \cap [\bar{\Gamma}])$.
- $S :=$ base scheme of α .

$$S = \emptyset \iff \alpha \text{ regular, } \mathcal{L} = \alpha^* \mathcal{O}(1), \widetilde{\mathcal{L}} = \pi^* \mathcal{L},$$

$$\text{a.p.p.} = \pi_*(ch(\pi^* \mathcal{L}) \cap [\bar{\Gamma}]) = ch(\mathcal{L}) \cap [W] = \exp(\ell) \cap [W].$$

Plan: If $S \neq \emptyset$, ‘correct’ fundamental class $[W]$ in this formula:

$$\text{a.p.p.} = \exp(\ell) \cap ([W] - \boxed{?}).$$

$$\begin{array}{ccc}
 & \bar{\Gamma} & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 W & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

Important remark: $\bar{\Gamma} \cong$ blow-up of W along S .

$E := \pi^{-1}(S)$ = exceptional divisor.

$E \cong \mathbb{P}(C_S W)$, the **Projective Normal Cone** of S in W .

Definition (PNC)

The **PNC** for a curve C is the projective normal cone of S in $W = \mathbb{P}^8$, i.e., E .

$[\pi^{-1}(S)]$ = class of PNC, in $A_*(\mathbb{P}^8 \times \mathbb{P}^{d(d+3)/2})$.

$$\begin{array}{ccc}
 & \bar{\Gamma} & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 W & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

- \mathcal{L} : line bundle on W , giving α ; $\tilde{\mathcal{L}} = \tilde{\alpha}^* \mathcal{O}(1)$.
- a.p.p. : $\pi_*(ch(\tilde{\mathcal{L}}) \cap [\bar{\Gamma}])$.
- $\ell := c_1(\mathcal{O}(\mathcal{L}))$, $h := c_1(\mathcal{O}(\tilde{\mathcal{L}}))$.
- $S :=$ base scheme of α ; $\bar{\Gamma} = Bl_S W$.
- $E = \pi^{-1}(S) =$ exceptional divisor = 'PNC'. $e := c_1(\mathcal{O}(E))$.
- Remark: $h = \ell - e$.
- $[E] = m_1[E_1] + \cdots + m_r[E_r]$, E_i irr. components, $m_i \in \mathbb{Z}^{\geq 0}$.
- Do I have all the notation I need? Not quite.

$$\begin{array}{ccc}
 & \bar{\Gamma} & \\
 \pi \swarrow & & \searrow \tilde{\alpha} \\
 W & \xrightarrow{\alpha} & \mathbb{P}^N
 \end{array}$$

- $\bar{\Gamma} = \text{Bl}_S W$; E = exceptional divisor = PNC.
- $\ell := c_1(\mathcal{O}(\mathcal{L}))$, $h := c_1(\mathcal{O}(\tilde{\mathcal{L}}))$, $e := c_1(\mathcal{O}(E))$. $h = \ell - e$.
- $[E] = m_1[E_1] + \cdots + m_r[E_r]$, E_i irr. components, $m_i \in \mathbb{Z}^{\geq 0}$.

Let

$$L_i := \sum_{k \geq 0} \frac{1}{k+1} \sum_{j=0}^k \frac{(-\ell)^{k-j}}{j!(k-j)!} \pi_*(h^j \cap [E_i])$$

Claim: These are the correction terms we were looking for.

Theorem

$$\begin{aligned}
 a.p.p. &:= \pi_*(ch(\tilde{\mathcal{L}}) \cap [\bar{\Gamma}]) \\
 &= \exp(\ell) \cap ([W] - (m_1 L_1 + \cdots + m_r L_r)) \quad \text{in } (A_* W)_{\mathbb{Q}}.
 \end{aligned}$$

$$[E] = m_1[E_1] + \cdots + m_r[E_r], \quad E_i \text{ irr. components, } m_i \in \mathbb{Z}^{\geq 0}$$

$$L_i := \sum_{k \geq 0} \frac{1}{k+1} \sum_{j=0}^k \frac{(-\ell)^{k-j}}{j!(k-j)!} \pi_*(h^j \cap [E_i])$$

Theorem

$$\begin{aligned} a.p.p. &:= \pi_*(ch(\widetilde{\mathcal{L}}) \cap [\bar{\Gamma}]) \\ &= \exp(\ell) \cap ([W] - (m_1 L_1 + \cdots + m_r L_r)) \quad \text{in } (A_* W)_{\mathbb{Q}}. \end{aligned}$$

This is the precise version of the claim that ‘the class of the PNC determines the a.p.p.’.

Proof: $h = \ell - e$.

$$\begin{aligned}
 \pi_*(ch(\widetilde{\mathcal{L}}) \cap [\bar{\Gamma}]) &= \pi_*(\exp(\ell - e) \cap [\bar{\Gamma}]) \\
 &= \exp(\ell) \cap \pi_*(\exp(-e) \cap [\bar{\Gamma}]) \\
 &= \exp(\ell) \cap ([W] - \pi_*(1 - \exp(-e)) \cap [\bar{\Gamma}])
 \end{aligned}$$

giving the correction term to the fundamental class as

$$\begin{aligned}
 \pi_*(1 - \exp(-e)) \cap [\bar{\Gamma}] &= \pi_* \sum_{i \geq 0} \frac{(-e)^i}{(i+1)!} \cap [E] \\
 &= \pi_* \sum_{i \geq 0} \frac{(h - \ell)^i}{(i+1)!} \cap (m_1[E_1] + \cdots + m_r[E_r]) \quad .
 \end{aligned}$$

Expand this expression to get the statement. □

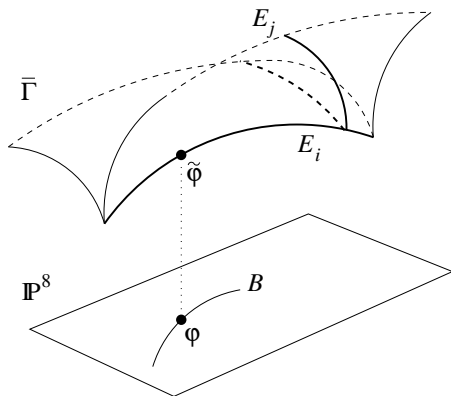
Bottom line:

- The a.p.p. may be computed by determining the irreducible components E_i of the PNC (i.e., the scheme $\pi^{-1}(S)$), and their multiplicities m_i .
- For each component E_i , get a contribution L_i in $A_*\mathbb{P}^8$, as a particular combination of push-forwards $\pi_*(h^j \cap [E_i])$.
- Then $\text{a.p.p.} = \exp(dH) \cap ([\mathbb{P}^8] - \sum m_i L_i)$.
- Why this is promising:
 - The action of $\text{PGL}(3)$ on \mathbb{P}^8 lifts to an action on $\bar{\Gamma}$, and each E_i is a dim. 7 union of orbits.
 - Each E_i dominates a union of small orbits.
 - The class of E_i may therefore be determined from a.p.p.s of curves with small orbits.
- **Need** tools to determine the E_i 's and the m_i 's explicitly.

(Naive) idea:

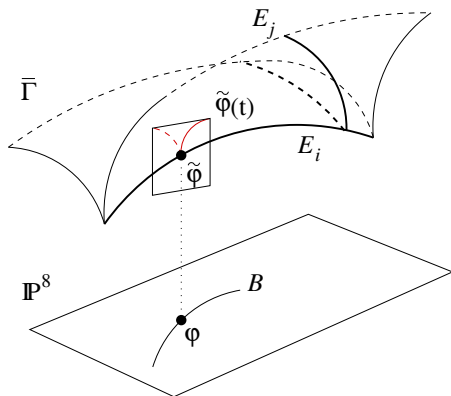
Have to catch all $\tilde{\varphi} \in E_i$, all i .

Such a $\tilde{\varphi}$ projects down to a $\varphi \in B = \text{base locus} = S_{\text{supp}}$.



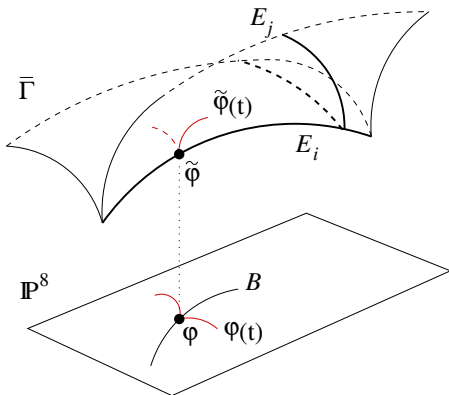
(Naive) idea:

Intersect $\bar{\Gamma}$ with a 7-dim nonsingular variety transversal to E_i in $\mathbb{P}^8 \times \mathbb{P}^N$.



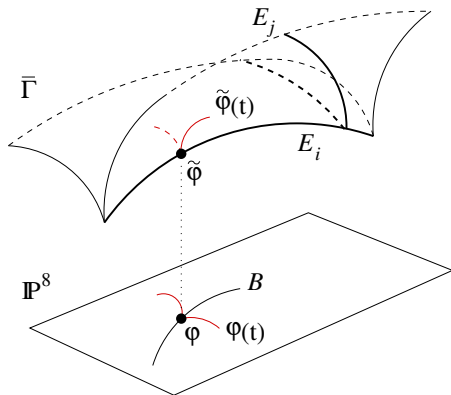
(Naive) idea:

This determines an arc $\tilde{\varphi}(t)$ through $\tilde{\varphi}$ in $\bar{\Gamma}$, projecting down to an arc $\varphi(t)$ in \mathbb{P}^8 through a point of B .



(Naive) idea:

Conversely, every $\tilde{\varphi}$ may be realized as $\tilde{\varphi}(0)$, where $\tilde{\varphi}(t)$ is the lift of a curve germ $\varphi(t)$ in \mathbb{P}^8 , with $\varphi(0) \in B$.



(Naive) idea:

Conversely, every $\tilde{\varphi}$ may be realized as $\tilde{\varphi}(0)$, where $\tilde{\varphi}(t)$ is the lift of a curve germ $\varphi(t)$ in \mathbb{P}^8 , with $\varphi(0) \in B$.

Set-theoretic description of components $E_i \rightsquigarrow$ classification of possible curve germs $\varphi(t)$ centered at points of B , in terms of their lifts to $\mathbb{P}^8 \times \mathbb{P}^{d(d+3)/2}$.

I.e., in terms of the corresponding **limits**:

$$t \neq 0 \quad : \quad \tilde{\varphi}(t) = (\varphi(t), C \circ \varphi(t)) \in \mathbb{P}^8 \times \mathbb{P}^{d(d+3)/2}$$
$$\tilde{\varphi} = \lim_{t \rightarrow 0} C \circ \varphi(t)$$

('Isotrivial flat completion problem')

Remarks:

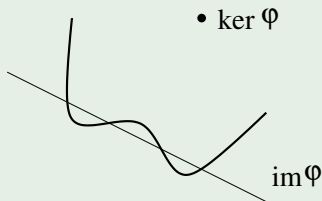
- The lim is only interesting if $\varphi = \varphi(0) \in B$, i.e., $\text{im } \varphi \subseteq C$.
- What about rk 2 matrices? If $\text{im } \varphi = \text{line} \not\subseteq C$, $C \circ \varphi = \text{'star'}$, α is defined at φ .

Example

$$C : 4x_0^4 - x_0^3x_2 - 5x_0^2x_1^2 + x_1^4 = 0$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix} : C \circ \varphi(t) : 4x_0^4 - x_0^3x_2t - 5x_0^2x_1^2 + x_1^4 = 0$$

$$t \rightarrow 0 : 4x_0^4 - 5x_0^2x_1^2 + x_1^4 = (x_0 - x_1)(x_0 + x_1)(2x_0 - x_1)(2x_0 + x_1)$$



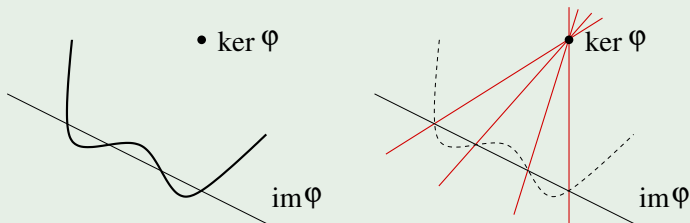
Limits?

Example

$$C : 4x_0^4 - x_0^3x_2 - 5x_0^2x_1^2 + x_1^4 = 0$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix} : C \circ \varphi(t) : 4x_0^4 - x_0^3x_2t - 5x_0^2x_1^2 + x_1^4 = 0$$

$$t \rightarrow 0 : 4x_0^4 - 5x_0^2x_1^2 + x_1^4 = (x_0 - x_1)(x_0 + x_1)(2x_0 - x_1)(2x_0 + x_1)$$



\rightsquigarrow **rank-2 limits.** They do not yield a component of the PNC.

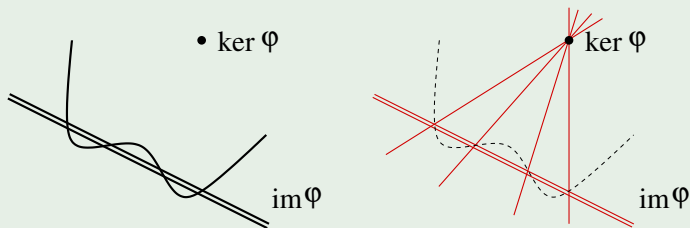
More interesting: $\text{rk } \varphi = 2$, and the line $\text{im } \varphi$ is a component of C . In this case, $C \circ \varphi \equiv 0$: α is *not* defined at φ .

Example

$$C : x_2^2(4x_0^4 - x_0^3x_2 - 5x_0^2x_1^2 + x_1^4) = 0$$

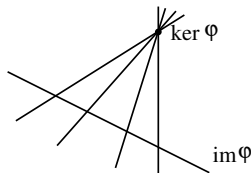
$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}, C \circ \varphi(t) : x_2^2 \textcolor{red}{t}^2(4x_0^4 - x_0^3x_2t - 5x_0^2x_1^2 + x_1^4) = 0$$

$$\lim_{t \rightarrow 0} : x_2^2(x_0 - x_1)(x_0 + x_1)(2x_0 - x_1)(2x_0 + x_1)$$



Type I components

Limits of this type (for $\text{rk } \varphi = 2$):



(we call them **fans**) have 7-dimensional orbit.

$\rightsquigarrow \{(\varphi, X) \mid \text{rk } \varphi = 2, \text{im } \varphi = \text{a component of } C, X \text{ a limit fan}\}$
fills up a component of the PNC, of 'type I'.

Type I components

There is a type I component of the PNC for every line $\subseteq C$.

Only contribution to the PNC from elements in B of rank 2.

$\text{rk } \varphi = 1$: $\varphi \in B \iff \text{im } \varphi = \text{a point } p \text{ of } C.$

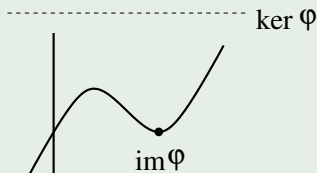
Nothing new unless p is on a *nonlinear* component C' of C .
 Else, p general: assume C' has equation $y = x^2 + \dots$ near p .

Example

$$C : (x_0^2 x_2 - x_0 x_1^2 - x_1^3)(x_0 + x_1) = 0$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}, C \circ \varphi(t) : \cancel{t}^2 (x_0^2 x_2 - x_0 x_1^2 - x_1^3 \cancel{t})(x_0 + x_1 \cancel{t}) = 0$$

$$\lim_{t \rightarrow 0} : x_0^2 (x_0 x_2 - x_1^2) = 0$$



$$\boxed{\text{rk } \varphi = 1} : \varphi \in B \iff \text{im } \varphi = \text{a point } p \text{ of } C.$$

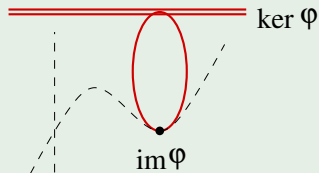
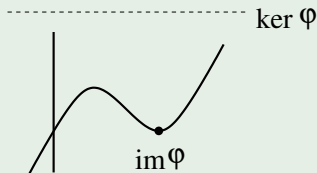
Nothing new unless p is on a *nonlinear* component C' of C .
 Else, p general: assume C' has equation $y = x^2 + \dots$ near p .

Example

$$C : (x_0^2 x_2 - x_0 x_1^2 - x_1^3)(x_0 + x_1) = 0$$

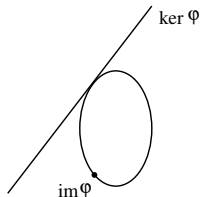
$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}, C \circ \varphi(t) : \cancel{t}^2 (x_0^2 x_2 - x_0 x_1^2 - x_1^3 \cancel{t})(x_0 + x_1 \cancel{t}) = 0$$

$$\lim_{t \rightarrow 0} : x_0^2 (x_0 x_2 - x_1^2) = 0$$



Type II components

Limit curves of this type:



have 6-dimensional orbit; plus one degree of freedom for $\text{im } \varphi \in C'$
 $\rightsquigarrow \dim\{(\varphi, X) \mid \text{rk } \varphi = 1, \text{im } \varphi \in C', X \text{ a conic+tangent line}\} = 7$
fill up components of the PNC, of 'type II'.

Type II components

There is a type II component of the PNC for every irreducible component of C of degree > 1 .

Global vs. Local components: III, IV, V

The situation gets more complicated now. . .

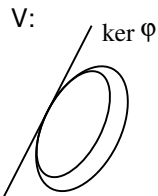
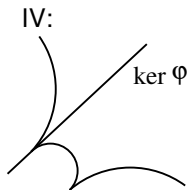
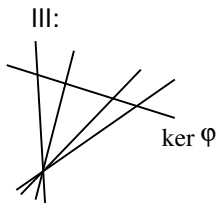
Type I+II depend on **global** features of C , i.e., its irreducible components.

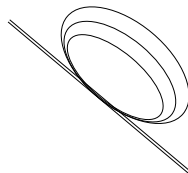
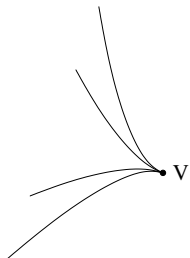
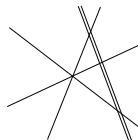
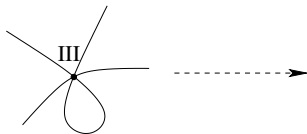
Other types arise because of **local** features of C :

they correspond to limits along $\varphi(t)$, for $\text{im } \varphi(0) = p \in C$, p **singular** or **inflectional** on C .

- **Type III**: Tangent cone to C at p is supported on ≥ 3 lines;
- **Type IV**: Determined by Newton polygon for C at p ;
- **Type V**: Det. by Puiseux pairs of formal branches of C at p .

Representative pictures for the limits:





Type III — example

Type III limits are straightforward.

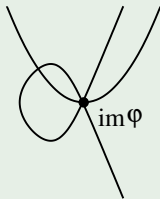
Example

$$C : (x_0x_2^2 - x_0x_1^2 - x_1^3)(x_0x_2 - x_1^2) = 0$$

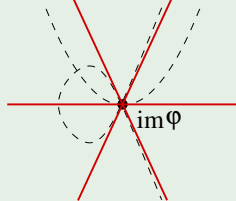
$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} : \textcolor{red}{t}^3(x_0x_2^2 - x_0x_1^2 - \textcolor{red}{t}x_1^3)(x_0x_2 - \textcolor{red}{t}x_1^2) = 0$$

$$t \rightarrow 0 : (x_0x_2^2 - x_0x_1^2)(x_0x_2) = x_0^2x_2(x_2 - x_1)(x_2 + x_1)$$

----- $\ker \varphi$



===== $\ker \varphi$



Type IV — example

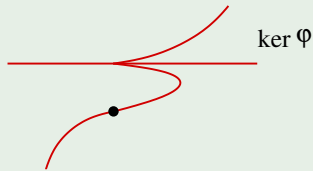
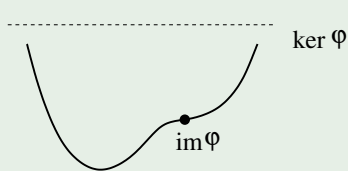
Type IV components arise already on nonsingular curves, due to inflection points.

Example

$$C : x_0^3 x_2 - x_0 x_1^3 - x_1^4 = 0$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^3 \end{pmatrix} : C \circ \varphi(t) : \textcolor{red}{t}^3 (x_0^3 x_2 - x_0 x_1^3 - \textcolor{red}{t} x_1^4) = 0$$

$$t \rightarrow 0 : x_0^3 x_2 - x_0 x_1^3 = x_0 (x_0^2 x_2 - x_1^3)$$



Type V — example

Type V components arise already on irreducible quartics.

Example

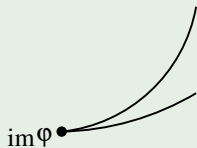
$$C : (x_1^2 - x_0 x_2)^2 - x_1^3 x_2 = 0$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^4 & t^5 & 0 \\ t^8 & 2t^9 & t^{10} \end{pmatrix} :$$

$$t^{20} (x_0^2 x_2^2 - 2x_0 x_1^2 x_2 + x_1^4 - x_0^4 - 5x_0^3 x_1 t - (9x_0^2 x_1^2 + x_0^3 x_2) t^2 + \cdots)$$

$$t \rightarrow 0 : x_0^2 x_2^2 - 2x_0 x_1^2 x_2 + x_1^4 - x_0^4 = (x_0 x_2 - x_1^2 + x_0^2)(x_0 x_2 - x_1^2 - x_0^2)$$

----- ker φ



----- ker φ



How to prove this is all?

Definition

Two germs $\varphi(t)$, $\psi(t)$ are **equivalent** if $\psi(t\nu(t)) \equiv \varphi(t) \circ m(t)$, with $\nu(t)$ a unit in $\mathbb{C}[[t]]$, and $m(t)$ a germ such that $m(0) = Id$.

Lemma

*C : plane curve. If $\varphi(t)$, $\psi(t)$ are equivalent germs, then $\lim_{t \rightarrow 0} C \circ \varphi(t) = \lim_{t \rightarrow 0} C \circ \psi(t)$.
It follows the lifts have the same center: $\tilde{\varphi}(0) = \tilde{\psi}(0)$.*

Proof: Let $F = 0$ be the equation of C .

Straightforward: If $\varphi(t)$, $\psi(t)$ are equivalent germs, then the initial terms in $F \circ \varphi(t)$, $F \circ \psi(t)$ coincide up to a nonzero multiplicative constant. □

Therefore, in probing E we may deal with germs up to equivalence.
The basic reduction is the following elementary observation:

Theorem

Every germ centered at φ , $\text{im } \varphi = p \in C$, is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

up to a parameter and coordinate change.

Here $1 \leq b \leq c$ and q, r, s polynomials such that $\deg(q) < b$, $\deg(r) < c$, $\deg(s) < c - b$, and $q(0) = r(0) = s(0) = 0$.

Proof: Linear algebra over $\mathbb{C}[[t]]$.



Strategy:

- Write equation for C at $p = (0,0)$ as

$$F(1, x_1, x_2) = F_m(x_1, x_2) + F_{m+1}(x_1, x_2) + \cdots + F_d(x_1, x_2) = 0$$

with $\deg F_i = i$, $F_m \neq 0$, $d = \deg C$.

F_m defines the tangent cone to C at p .

- Examine $\lim_{t \rightarrow 0} F \circ \varphi(t)$ with

$$\begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

- Eliminate all cases leading to 'rank-2 limits' (stars).
- See what's left!

$$q = r = s = 0$$

If $q(t) = r(t) = s(t) = 0$, the germ is a **1-PS**:

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}. \text{ May assume } b, c \text{ are relatively prime.}$$

- $b = c = 1 \rightsquigarrow$ **Type III**.
- Else...

Lemma

If $b < c$ and $x_2 \nmid F_m$, then $\lim_{t \rightarrow 0} C \circ \varphi(t)$ is a rank-2 limit.

Lemma

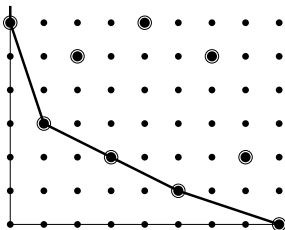
If $b < c$ and $x_2 \mid F_m$, and $-b/c$ is not a slope of the Newton polygon for C at p , then $\lim_{t \rightarrow 0} C \circ \varphi(t)$ is supported on a triangle.

Neither case contributes components to PNC.

- $b < c$, $x_2|F_m$, $-b/c = \text{slope of the Newton polygon strictly between } -1 \text{ and } 0$.

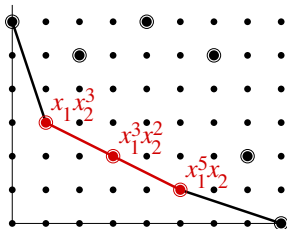
Claim

These germs lead to components of type IV.



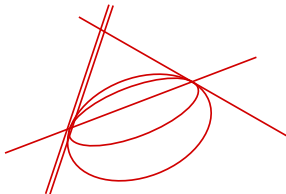
Slopes between -1 and 0 : $-1/2$, $-1/3$

Slope $= -1/2 \rightsquigarrow \varphi(t)$ with $b = 1$, $c = 2$



$$\lim_{t \rightarrow 0} C \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix} : ax_1x_2^3 + bx_1^3x_2^2 + cx_1^5x_2^2$$

I.e.: $x_0^{d-6}x_1x_2(x_1^2 + \lambda x_0x_2)(x_1^2 + \mu x_0x_2)$, **Type IV**



q, r, s not all 0

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}, \text{ with } q, r, s \text{ not identically 0.}$$

This is much, much subtler! Key reduction:

Lemma

q, r, s not identically 0. If $\lim_{t \rightarrow 0} C \circ \varphi(t)$ is not a rank-2 limit, then C has a formal branch $x_2 = f(x_1)$, tangent to $x_2 = 0$, such that φ is equivalent to a germ

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix},$$

with $a < b < c$ positive integers.

Underlining means: truncation modulo t^c .

Formal branch?

Local equation for C at $p = (1 : 0 : 0)$: $\Phi(x, y) = F(1 : x : y) = 0$.

- Decompose $\Phi(x, y)$ in $\mathbb{C}[[x, y]]$: $\Phi = \prod_i \Phi_i$, Φ_i irreducible.
- Weierstrass preparation: $\Phi_i \in \mathbb{C}[[x]][y]$ (up to a unit).
- Then Φ_i splits as a product of linear factors over $\mathbb{C}[[x^*]]$:

$$\Phi_i(x, y) = \prod_{j=1}^{m_i} (y - f_{ij}(x))$$

$f_{ij}(x)$ a power series with **rational** nonnegative exponents:
Puiseux series, exponents.

- This gives the ‘formal branches’ $y = f(x)$ for C at p .

Type V components are detected by germs

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}, \text{ where } f \text{ is a formal branch for } C \text{ at } p.$$

Further subtlety: Type V limits ('quadritangent' conics) only arise when there exist two branches that agree modulo x^γ , differ at x^γ , for some γ larger than the order of the branches.

Example

$$C : x_0(x_0x_2 - x_1^2)^2 = x_1^5$$

$$(x_0 : x_1 : x_2) = (1 : x : y) : (y - x^2)^2 = x^5 \text{ i.e.: } y - x^2 = \pm x^{5/2}.$$

Two formal branches: $y = x^2 - x^{5/2}$, $y = x^2 + x^{5/2}$.

Situation as described above, with $\gamma = \frac{5}{2}$.

'Reason' for γ : need ≥ 2 branches to differ, to get ≥ 2 quadritangent conics, $\dim \text{ orbit} = 7$, contribution to PNC.

Example

$$y = f(x) = x^2 \pm x^{5/2}$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}, \quad f'(t^a) = 2t^a \pm \dots$$

Fine print: $a = 4$, $b = 5$, $c = 10$. (So $c/a = 5/2 = \gamma$)

$$\rightsquigarrow \varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^4 & t^5 & 0 \\ t^2 & 2t^9 & t^{10} \end{pmatrix}$$

$$\lim_{t \rightarrow 0} C \circ \varphi(t) = x_0(x_0x_2 - x_1^2 + x_0^2)(x_0x_2 - x_1^2 - x_0^2)$$

$\ker \varphi$



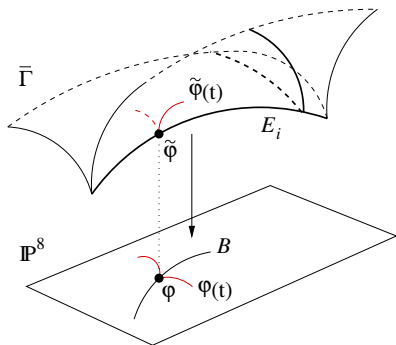
This concludes the set-theoretic description of the PNC. Summary:

- Five ‘types’ of components, 2 global and 3 local:
 - Type I: From linear components of C .
 - Type II: From nonlinear components of C .
 - Type III: From points with ≥ 3 lines in the tangent cone.
 - Type IV: From sides of Newton polygon.
 - Type V: From formal branches.

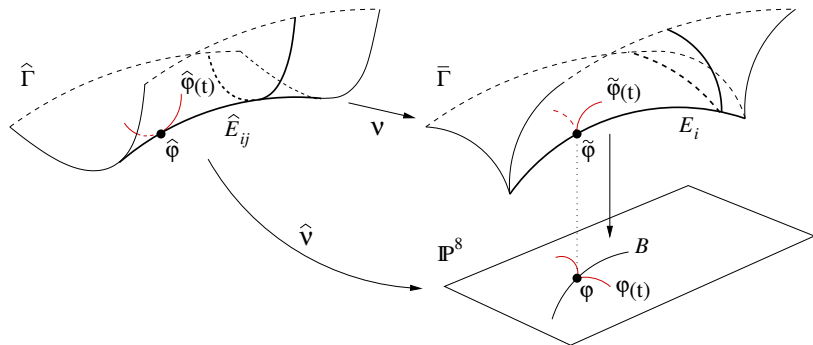
We are not done!

We need the PNC *as a cycle*, i.e., we need the multiplicity of each component.

The idea in a picture:



The idea in a picture:



Consider the **normalization** $\nu : \hat{\Gamma} \rightarrow \bar{\Gamma}$.

$E = \sum m_i E_i$ pulls back to $\hat{E} = \sum m_{ij} \hat{E}_{ij}$.

Lemma

$m_i = \sum_j e_{ij} m_{ij}$, where $e_{ij} = \text{degree of } \hat{E}_{ij} \rightarrow E_i$.

- e_{ij} : obtained by comparing stabilizer subgroups.
- m_{ij} : minimum 'weight' of a germ $\varphi(t)$.

e_{ij} : $\text{PGL}(3)$ acts on both $\bar{\Gamma}$ and $\hat{\Gamma}$.

For $\hat{\varphi}$, $\tilde{\varphi} = \nu(\hat{\varphi})$, $\text{Stab } \hat{\varphi}$ is a subgroup of $\text{Stab } \tilde{\varphi}$.

Lemma

The degree of \hat{E}_{ij} over E_i is the index of $\text{Stab } \hat{\varphi}$ in $\text{Stab } \tilde{\varphi}$.

Lemma

The degree of \hat{E}_{ij} over E_i is the index of $\text{Stab } \hat{\varphi}$ in $\text{Stab } \tilde{\varphi}$.

Reason (III, IV, V): Both E_i , \hat{E}_{ij} are closures of $\text{PGL}(3)$ -orbits.

$\text{Stab } \tilde{\varphi}$: Known from classification of small orbits.

$\text{Stab } \hat{\varphi}$: May be realized as the subgroup of $\text{Stab } \tilde{\varphi}$ consisting of automorphisms induced on $\tilde{\varphi}(0)$ by a reparametrization of $\tilde{\varphi}(t)$. \square

Type IV: number of roots of 1 preserving the tuple in the limit.

Type V: more 'interesting'. (Look at the paper for details!)

m_{ij} :

Lemma

If $F = 0$ is the equation for C , $m_{ij} =$ minimum order of vanishing of $F \circ \varphi(t)$ for a germ $\varphi(t)$ s.t. $\tilde{\varphi}$ is a general point of E_i .

Proof: $m_{ij} = \min$ order of vanishing of $\hat{\nu}^{-1}(S) \cdot \hat{\varphi}(t)$
= min order of vanishing of restriction of equation for S to $\varphi(t)$.

S : cut out by point-conditions $F(p) = 0$, $p \in \mathbb{P}^2$.

Restrict to $\varphi(t)$:

$$F \circ \varphi(t) = t^m G(x_0 : x_1 : x_2) + t^{m+1} H(x_0 : x_1 : x_2) + \cdots, \quad G \not\equiv 0$$

$$G(x_0 : x_1 : x_2) = \text{equation of } X := \lim_{t \rightarrow 0} C \circ \varphi(t)$$

$$p = (x_0 : x_1 : x_2) \notin X \rightsquigarrow m = \min \text{ order of vanishing.}$$

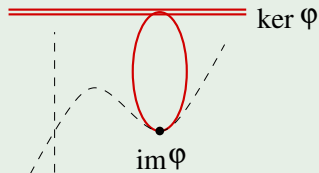
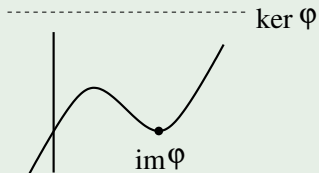


Example

$$C : (x_0^2 x_2 - x_0 x_1^2 - x_1^3)(x_0 + x_1) = 0$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}, C \circ \varphi(t) : \boxed{t^2} (x_0^2 x_2 - x_0 x_1^2 - x_1^3 t)(x_0 + x_1 t) = 0$$

$$\lim_{t \rightarrow 0} : x_0^2 (x_0 x_2 - x_1^2) = 0$$

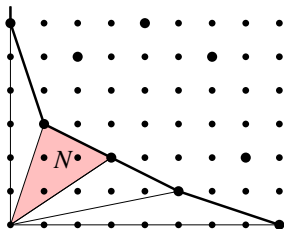


Order of vanishing = 2 \rightsquigarrow mult. of corr. type II component = 2.

Example: Multiplicity for type IV components

Multiplicities for type IV

Type IV: from sides of Newton polygon.



N = one segment of the triangle determined by the side.

Multiplicities for type IV

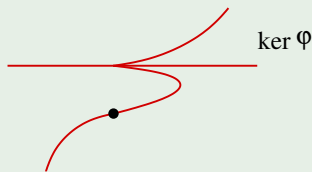
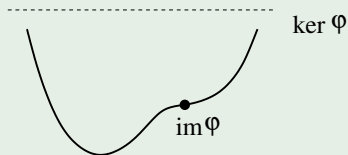
Mult. = $2 \cdot (\text{area of } N) \cdot \#\{\text{roots of 1 preserving tuple in the limit}\}$

Example

$$C : x_0^3 x_2 - x_0 x_1^3 - x_1^4 = 0$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^3 \end{pmatrix} : C \circ \varphi(t) : \cancel{t}^3 (x_0^3 x_2 - x_0 x_1^3 - \cancel{t} x_1^4) = 0$$

$$t \rightarrow 0 : x_0^3 x_2 - x_0 x_1^3 = x_0 (x_0^2 x_2 - x_1^3)$$

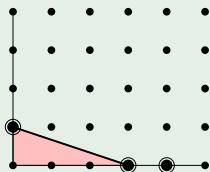


Example

$$C : x_0^3 x_2 - x_0 x_1^3 - x_1^4 = 0$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^3 \end{pmatrix} : C \circ \varphi(t) : \boxed{t^3} (x_0^3 x_2 - x_0 x_1^3 - t x_1^4) = 0$$

$$t \rightarrow 0 : x_0^3 x_2 - x_0 x_1^3 = x_0 (x_0^2 x_2 - x_1^3)$$



$$2 \cdot \text{Area} = 3$$

Example: general nonsingular curve

C : nonsingular, degree d , $3d(d-2)$ ordinary flexes.

The PNC consists of:

$$E_G = \{(\varphi, X) \mid \text{im } \varphi \in C, \\ X = \text{conic tangent to } \ker \varphi \text{ union } (d-2)\text{-fold kernel line}\}$$

$$E_F = \{(\varphi, X) \mid \text{im } \varphi = \text{a flex of } C, X = \text{cuspidal cubic} \\ \text{with cusp tangent on } \ker \varphi \text{ union } (d-3)\text{-fold kernel line}\}$$

As a cycle: $2[E_G] + 3[E_F]$. *Correction terms* in \mathbb{P}^8 :

$$[L_G] = \frac{6dH^5}{5!} - \frac{4d(5d+18)H^6}{6!} + \frac{12d(9d+8)H^7}{7!} - \frac{6720d^2H^8}{8!}$$

$$[L_F] = \frac{5H^6}{6!} - \frac{72H^7}{7!} + \frac{591H^8}{8!}$$

$$\rightsquigarrow \text{a.p.p.} = \exp(dH) \cap ([\mathbb{P}^8] - 2[L_G] - 3[L_F])$$

\rightsquigarrow same result as by blow-ups.

Adjusted predegree polynomials for **arbitrary** curves may be computed in this way.

Raw expressions are difficult to handle.

But they lead to manageable formulas for, e.g., unibranched singularities: 'reasonable' expression in terms of Puiseux pairs.

Multibranched singularities: Probably also possible to simplify results. Nice open question. . .

Example

$p \in \mathbb{C}$: Puiseux expansion

$$\begin{cases} z = (a_n t^n + \cdots +) a_{e_1} t^{e_1} + \cdots + a_{e_r} t^{e_r} \\ y = t^m \end{cases}$$

with $m < n \leq e_1 < \cdots < e_r$.

\rightsquigarrow 'simple' formula for contribution of p in terms of m , n , and the exponents e_i . For instance, if the e_i 's are not there,

$$mn \left\{ \left(P(m, 2m) - P(m, n) \right) \cdot \left(\frac{k^2 H^6}{6!} + \frac{k H^7}{7!} + \frac{H^8}{8!} \right) \right\}_2$$

where $\{\cdot\}_2 = \text{coeff. of } k^2$, and

$$P(a, b) = \frac{a^2 b^2}{(1 + ak)^3 (1 + bk)^3} - \frac{4}{(1 + k)^3 (1 + 2k)^3}$$

Back to p. 31! “One case when they are reasonably neat:
contribution of unbranched singularities. . .”

C of degree d , ordinary flexes, and singularities of type (t^m, t^n) ,
with no further Puiseux pairs. Also assume $\text{Stab } C$ is trivial.

Theorem

The degree of the orbit closure of C is

$$d^8 - \left\{ (1 + dk)^8 \left[\frac{4d^2}{(1 + k)^3(1 + 2k)^3} \right. \right. \\ \left. \left. + \sum_{p \in C \text{ of type } (t^m, t^n)} mn \left(\frac{m^2 n^2}{(1 + mk)^3(1 + nk)^3} - \frac{4}{(1 + k)^3(1 + 2k)^3} \right) \right] \right\}_2$$

(Here $\{\cdot\}_2$ extracts the coefficient of k^2 in the given expression.)

THANK
YOU

for your attention