Minimal Model Program on q.polarized varieties

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For instance let $X \subset \mathcal{P}^N$ be a projective variety and $L := \mathcal{O}(1)_{|X}$, or better its (partial) desingularizaton and the pull back of L.

Classical problems

Problem Given a general element $D \in |L|$
(assume that X is not a cone over D).

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Sommese proved that abelian and bi-elliptic surfaces cannot be ample sections.

Adjunction Theory

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Assume that there exist r sections of $|L|$ which intersect in a $n-r$ variety D, with terminal singularities.

To get nefness of $K_X + rL$ implies, by adjunction $(K_X + rL)_{|D} = K_D$, to get a minimal model for D.

zip L into a boundary

Let (X, L) be a quasi-polarized variety and $r \in \mathbb{Q}^+$.

Lemma. There exists an effective $\mathbb{Q}$-divisor Δ^r on X such that

$$rL \sim_{\mathbb{Q}} \Delta^r \quad \text{and} \quad (X, \Delta^r) \text{ is Kawamata log terminal.}$$

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Def. A log pair (X, Δ), i.e a normal variety X and an effective $\mathbb{R}$ divisor Δ, is Kawamata log terminal (klt) if

- $K_X + \Delta$ is $\mathbb{R}$-Cartier
- for a (any) log resolution $g : Y \longrightarrow X$ we have $g^*(K_X + \Delta) = K_Y + \Sigma b_i \Gamma_i$ with $b_i < 1$, for all i.

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Proof: For a Cartier divisor L, nef and big is equivalent to the existence of $E > 0$ and A_k, $\mathbb{Q}$-ample divisor, such that $L \sim A_k + (1/k)E$ for $k >> 0$.

Minimal Model Program- BCHM

By BCHM on a klt log pair (X, Δ) we can run a

$K_X + \Delta$- Minimal Model Program with scaling:

$$(X_0, \Delta_0) = (X, \Delta) \to (X_1, \Delta_1) \to - - - -- \to (X_s, \Delta_s)$$

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1) (X_i, Δ_i) is a klt log pair, for $i = 0, ..., s$;

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3) either $K_{X_s} + \Delta_s$ is nef ((X_s, Δ_s) is a log Minimal Model), or $X_s \to Z$ is a Mori fiber space relatively to $K_{X_s} + \Delta_s$ (depending on the pseudeffectivity of $K_X + \Delta$).

MMP for a q.p. pair

Let (X, L) be a quasi-polarized variety. Take $r \in \mathbb{Q}^+$ and let (X, Δ^r) be the klt log pair such that $rL \sim_{\mathbb{Q}} \Delta^r$.

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Run a $K_X + \Delta^r$-MMP and get a birational klt pair (X_s, Δ^r_s) which is either a log Minimal Model (i.e. $K_{X_s} + \Delta_s$ is nef), or $X_s \to Z$ is a Mori fiber space relatively to $K_{X_s} + \Delta^r_s$.

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Remarks/Problems

- (X_s, Δ^r_s) is not necessarily an (r) q.p. pair, i.e. we do not have a priori a nef and big Cartier divisor L_s such that $rL_s \sim_{\mathbb{Q}} \Delta^r_s$.
- Beyond the existence of the MMP, it would be nice to have a "description" of each steps and eventually of the Mori fiber spaces.

Extremal rays

For the above program we study the rays $R = \mathbb{R}^+[C] \in \overline{NE(X)}_{(K_X+rL)<0} \subset \overline{NE(X)}_{K_X<0}$.

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Note that $\tau > r$.
Note also that the contraction φ is supported by the divisor $K_X + \tau L$

Base point free technique

Theorem
[..., Fano, Fujita, Kawamata, Kollar, A-Wisniewski, Mella, A-Tasin, ...]

Let X, $R = \mathbb{R}^+[C]$, $\varphi : X \to Y$, F and L as above; assume that $L \cdot C > 0$.

- $\dim F \geq \tau - 1 > r - 1$; if φ is birational $\dim F \geq \tau > r$.
- If $\dim F < \tau + 1$, or $\dim F \leq \tau + 1$ if φ is birational, then L is very ample (relatively to φ).
- If $\dim F < \tau + 3$ then there exists $X' \in |L|$ with "good" singularities (i.e. as in X).

$r = (n-1)$

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1) $L_i C_i = 0$, otherwise, by the above Theorem, we have the contradiction $(n-1) \geq \dim F > r \geq (n-1)$.
2) Let $\varphi_i : X_i \to Y$ be the contraction associated with R_i. We have a Cartier divisor L'_{i+1} such that $\varphi^*(L'_{i+1}) = L_i$.
If φ is birational $(X_{i+1}, L_{i+1}) := (Y, L'_{i+1})$, if φ is small $(X_{i+1}, L_{i+1}) := (X_i^+, \varphi^+(L'_{i+1}))$, where $\varphi^+ : X_i^+ \to Y$ is the flip.

the zero reduction

Proposition. Given a q.p. pair (X, L) it is possible to run a MMP which contracts all extremal rays on which L is zero and obtain a q.p. pair (X', L') which is birational equivalent to (X, L) and such that:

- either $K_{X'} + (n-1)L'$ is nef
- or (X', L') is a Mori space relative to $K_{X'} + (n-1)L'$ and L' is a (relatively) very ample Cartier divisor.

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By very classical results in the second case the q.p. pair (X', L') is in a obvious finite list of examples: $(\mathbb{P}^n, \mathcal{O}(1))$, $(Q, \mathcal{O}(1))$, scrolls, del Pezzo.

Applications

Let (X, L) be a quasi-polarized variety and $g(X, L)$ be its sectional genus: $2g(X, L) - 2 = (K_X + (n-1)L) \cdot L \cdot \ldots \cdot L)$.

(if L is spanned it is the genus of a curve intersection of $n - 1$ general elements in $|L|$.

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- Classification of pairs with $g(X, L) = 1$.

- Classification of pairs of minimal degree
(i.e. $L^n = h^0(X, L) - n$).

Castelnuovo-Kawakita

Let $\varphi : X \to Y$ be a birational contraction associated with an extremal ray $R = \mathbb{R}^+[C]$ on a q.p. pair, such that $L{\cdot}C > 0$ and $\tau(X, L) > (n-2)$. (These are the birational maps in a $K_X + \Delta^{n-2}$-MMP)

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Theorem. $\varphi : X \to Y$ is the weighted blow-up of a smooth point in Y of weights $(1, 1, b, ..., b)$, where b is a natural positive number.
$L' = \varphi_*(L)$ is a Cartier divisor on Y such that $\varphi^* L' = L + bE$, where E is the exceptional (Weil) divisor.
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Definition. φ is called a Castelnuovo-Kawakita contraction.

Proof

We have $(n-1) \geq \dim F > (n-2)$, i.e. $\dim F = (n-1) \leq \tau + 1$. Therefore φ is a contraction of a divisor to a point and we can assume L is very ample.

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Horizontal slicing: Let $X' \in |L|$ be a generic divisor: we have:
$\varphi_{|X'} := \varphi' : X' \to S'$ is a contraction with connected fibre, around $F \cap X'$, supported by $K_{X'} + (\tau - 1)L_{|X'}$.

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By horizontal slicing with $(n-2)$ general sections of $|L|$ we can reduce to the surface case and $\tau > 0$. Surfaces with terminal singularities are smooth. Apply now Castelnuovo's Theorem to have that the (surface) image is smooth .

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Since Y has terminal singularities this implies that Y is smooth at the exceptional point.

Proof

Since $X = \mathrm{Proj}_{\mathcal{O}_Z}(\oplus_{d \geq 0} f_*(\mathcal{O}_X(-dbE))$,
we need to prove that

$$f_*(\mathcal{O}_X(-dbE) = (x_1^{s_1} \cdots x_n^{s_n} \mid s_1 + s_2 + \sum_{j=3}^{n} bs_j \geq db) := I^d$$

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Induction:
a) consider a general element $X' \in |L|$ such that $f_*(X') = \{x_n = 0\}$ (since Y is smooth, $f_*(X')$ is Cartier).The restricted morphism $f' := f_{|X'} : X' \to Z'$ is a divisorial contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$.

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b) The exact sequence
$0 \to \mathcal{O}_X(-L - dbE) \to \mathcal{O}_X(-dbE) \to \mathcal{O}_{X'}(-dbE) \to 0$
and the Relative Kawamata-Viehweg Vanishing gives

$$0 \to f_*\mathcal{O}_X(-(d-1)bE) \xrightarrow{\cdot x_n} f_*\mathcal{O}_X(-dbE) \to f_*\mathcal{O}_{X'}(-dbE) \to 0.$$

By induction on n and d the proposition follows

First Reduction

Proposition-Part 1. Let (X, L) be a q.p. pair. There exists a q.p. pair (X'', L'') which is a $(K_X + \Delta^{n-2})$-MM and which can be obtained with the following procedure:

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Definition The pair (X'', L'') is called a First Reduction of the pair (X, L).

First Reduction

Proposition-Part 2. Let (X, L) be a q.p. pair and let (X'', L'') be its First Reduction. Then

- either $K_{X''} + (n-2)L''$ is nef
- or $X'' \to Z$ is a Mori fiber space relatively to $K_{X''} + (n-2)L''$ and L'' is (relatively) very ample with one exception (del Pezzo manifold). In all cases there exists a divisor in $|L''|$ with good singularities.

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Proposition-Part 2. Let (X, L) be a q.p. pair and let (X'', L'') be its First Reduction. Then

- either $K_{X''} + (n-2)L''$ is nef
- or $X'' \to Z$ is a Mori fiber space relatively to $K_{X''} + (n-2)L''$ and L'' is (relatively) very ample with one exception (del Pezzo manifold). In all cases there exists a divisor in $|L''|$ with good singularities.

Remark. The classification of the pairs in the second part, thank to the existence of a good section, is classical and reduces to the theory of algebraic surfaces. (Quadric fibration, del Pezzo manifolds, ...)

Applications

Theorem. Let $Y \subset \mathbb{P}^N$ be a non degenerate projective variety of dimension $n \geq 3$ of degree d and let $\tilde{L} := \mathcal{O}(1)_{|Y}$. Assume that $d < 2codim_{\mathbb{P}^N}(X) + 2$.

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Then on a desingularization (X, L) the divisor $K_X + (n-2)L$ is not pseudoeffective.

Therefore $(Y, \mathcal{O}(1))$ is equivalent, via birational equivalence and first-reduction, to a q.p. pair (X'', L'') in the above Remark.

Kawakita

Let $\varphi : X \to Y$ be a birational contraction associated with an extremal ray $R = \mathbb{R}^+[C]$ on a q.p. pair, such that $L \cdot C > 0$ and $\tau(X, L) > (n-3)$. (These are the birational maps in a $K_X + \Delta^{n-3}$-MMP)

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Theorem. Then f is a weighted blow-up of type $(1, a, b, c, \ldots, c)$, where a, b, c are positive integers, $(a, b) = 1$ and $ab|c$.
$L' = \varphi_*(L)$ is a Cartier divisor on Y such that $\varphi^* L' = L + cE$, where E is the exceptional (Weil) divisor.
In particular $\tau(X, L) = (n - 2 + 1/c) > (n-2)$.

Proof

Since $X = \mathrm{Proj}_{\mathcal{O}_Z}(\oplus_{d\geq 0} f_*(\mathcal{O}_X(-dcE))$,
we need to prove that

$$f_*(\mathcal{O}_X(-dcE) = (x_1^{s_1}\cdots x_n^{s_n} \mid s_1 + s_2a + s_3b + \sum_{j=4}^{n} cs_j \geq dc) := I^d$$

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Induction:
a) consider a general element $X' \in |L|$ such that $f_*(X') = \{x_n = 0\}$ (since Y is smooth, $f_*(X')$ is Cartier).The restricted morphism $f' := f_{|X'} : X' \to Z'$ is a divisorial contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$.

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b) The exact sequence
$0 \to \mathcal{O}_X(-L - dcE) \to \mathcal{O}_X(-dcE) \to \mathcal{O}_{X'}(-dcE) \to 0$
and the Relative Kawamata-Viehweg Vanishing gives

$$0 \to f_*\mathcal{O}_X(-(d-1)cE) \xrightarrow{\cdot x_n} f_*\mathcal{O}_X(-dcE) \to f_*\mathcal{O}_{X'}(-dcE) \to 0.$$

By induction on n and d the proposition follows

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Theorem. Then, outside a finite set of points, C is a smooth and contained in the smooth locus of Y.
φ is the blow up of an ideal $I^{(n)}$ which is the symbolic power of an ideal I which is supported along C and, outside these finite points, is the ideal of the weighted blow up of a smooth manifold along a smooth curve with weight $(1, 1, b, ..., b, 0)$, b a positive integer.

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Theorem. Then .... ???